Conic optimization: the past, the present and the future

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Outline

Optimization problems

optimize *objective(s)*

s.t. *constraints are satisfied*

Objectives

- cost, return
- risk
- error
- SNR
- time
- heat
- lift, drag
- ...

Constraints

- laws of nature
- manufacturing restrictions
- standards
- design requirements
- resources
- tolerances
- logical relationships
- ...

Elements of optimization

- (Decision/design) variables x, y, s
  - continuous
  - discrete
  - binary
- Problem data (parameters) A, b, c, f, g
- Constraints/objectives $f(x) \overset{=}{\leq} 0$
  - linear
  - quadratic
  - nice nonlinear (convex)
  - ugly nonlinear (nonconvex)
- Feasible solution: satisfies all the constraints
- Typical problem

$$\begin{aligned} \min\ f(x) \\ Ax = b \\ g(x) \leq 0 \\ h(x) = 0 \end{aligned}$$

1. Optimization problems in general
2. Motivating examples for conic optimization
    - Stability analysis and eigenvalue problems
    - Combinatorial optimization
    - Optimization under uncertainty
3. Conic optimization
4. The future of conic optimization
5. Other activities

Stability analysis

Theorem (Lyapunov stability)

Trajectories of $\dot{x} = Ax$ converge to 0 if and only if there is a positive definite P such that $A^T P + PA$ is negative definite.

Theorem (Circle criterion)

Linear state-feedback system:

$$\dot{x} = Ax + Bw, \quad x(0) = x_0$$
$$v = Cx$$

$$\sigma(v, w) = (\beta v - w)^T (w - \alpha v) \geq 0, \quad \textit{(sector constraint, } \alpha < \beta\textit{)}$$

Equivalent condition for quadratic stability:

$$P \succ 0$$
$$\begin{pmatrix} A^T P + PA - 2\beta\alpha C^T C & PB + (\beta + \alpha) C^T \\ B^T P + (\beta + \alpha) C & -2 \end{pmatrix} \prec 0$$

Eigenvalue optimization

- Looking for a "good" positive definite matrix

Problem

Given matrices $A_1, \ldots, A_m$ find a nonnegative linear combination whose smallest eigenvalue is maximal.

- SDP formulation

$$\max \lambda$$

$$\sum_{i=1}^{m} A_i y_i - \lambda I \text{ is PSD}$$

$$y_i \geq 0,\ i = 1, \ldots, m$$

- Needed for effective stability analysis

Relaxation of binary variables

- Binary constraints: $x_i \in \{0, 1\}$
- LP relaxation: $x_i \in [0, 1]$, easy, but too weak
- Equivalent form:

$$z_i = 2x_i - 1$$
$$z_i^2 = 1 (\Leftrightarrow z_i = \pm 1)$$

- Matrix form ($Z = zz^T$):

$$Z \text{ is PSD}$$
$$\operatorname{diag}(Z) = 1$$
$$\operatorname{rank}(Z) = 1 (\Leftrightarrow Z = zz^T)$$

- SDP relaxation is stronger than the LP relaxation

Optimization under uncertainty

- Sources of uncertainty
  - measurement errors
  - (random) noise
  - unpredictability, future
  - vibration
  - technological limits
  - material imperfections
  - computer round-off errors
- Application areas
  - medical treatment planning
  - structural design
  - robust portfolio selection

Solution approaches

- Scenario enumeration
  - include all the possible values of a_i
  - possibly infinitely many...
  - take a subset - column generation
- Deterministic approach
  - put a threshold on the possible error
  - rewrite the problem - if possible
- Probabilistic approach
  - assume a distribution for the uncertainty (Gaussian)
  - ensure the constraint is satisfied with high probability

Example - Robust linear programming

- Standard linear programming

$$\min c^T x$$
$$a_i^T x \geq b_i,\ i = 1, \ldots, m$$

- Assume the data is uncertain: $a_i \sim N(\bar{a}_i, \Sigma_i)$
- We want $P(a_i^T x \geq b_i) \geq \eta$, which is equivalent to

$$\left\| \Sigma_i^{1/2} x \right\|_2 \leq \Phi(\eta) \left(b_i - \bar{a}_i^T x \right)$$

- Assuming $a_i \sim U(\bar{a}_i - \beta_i, \bar{a}_i + \beta_i)$ leads to an LP
- In general, robustification increases complexity

Other applications

- Quadratic programming
- Quadratic linear fractions

$$\sum_{i=1}^{r} \frac{\|A_i x + b_i\|^2}{a_i^T x + \beta_i} \leq t$$

- Inequalities with rational powers

$$x_1^{-5/6} x_2^{-1/3} x_3^{-1/2} \leq t$$
$$x_1, x_2, x_3 \geq 0$$

- Structural/shape design
- Sensor network localization
- Sum-of-squares optimization
- Minimization of univariate polynomials

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2. Motivating examples for conic optimization
3. Conic optimization
   - Theoretical background
   - Software tools
4. The future of conic optimization
5. Other activities

Symmetric conic optimization in general

- The primal-dual problems

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\in \mathcal{K} \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\in \mathcal{K}, \end{aligned}$$

- The cone $\mathcal{K}$ can be

  Linear: $x \geq 0$

  Second-order: $x_0 \geq \|x\|_2$

  Rotated second-order: $x_0 x_1 \geq \|x_{2:n}\|$, and $x_0 \geq 0$

  Semidefinite: x is (can be assembled into) a symmetric, positive semidefinite matrix, or a

  product of these.

Solving conic optimization

- Algorithms: mostly interior point methods
- Iterations
  - SDP: $\mathcal{O}(\sqrt{n})$
  - SOCP: $\mathcal{O}(\sqrt{\#\text{cones}})$
  - practice: $\approx 50 - 100$
- Cost of one iteration
  - SDP: $\mathcal{O}(mn^3 + m^2n^2 + m^3)$
  - SOCP: $\mathcal{O}(m^3 + km^2n^2)$
  - much less for sparse data
- Problem sizes:
  - SDP: $m \leq 10000$, $n \leq 5000$
  - SOCP $m \leq 10000$, $k \leq 10000$
  - more with sparse data
- High accuracy (10^{-6})

Software

- Solvers - mostly open source, last 10 years

  SOCP: MOSEK, SeDuMi, LOQO, SDPT3, PENNON

  SDP: CSDP, SDPA, SeDuMi, SDPT3, PENSDP

- Modeling languages
  - No commercial support for SDP
  - Scarce support for SOCP
  - Only academic packages (Yalmip, CVX)

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   - Theory and algorithms
   - Applications and software
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Theory and algorithms

- More efficient algorithms
  - IPMs have reached their limit
  - simplex-like algorithm
  - better handling of sparsity for SDP
  - preprocessing
- More general cones
  - homogeneous cones
  - nonnegative polynomials
- Cone intersections
  - PSD matrix with nonnegative numbers
  - intersection of second-order cones
- Integer conic programming
  - very limited theory/algorithms
  - efficient cut generation is an open problem
- Rounding procedure
  - getting an exact solution under some conditions

Applications and software

- Reliability
  - too many solver failures
  - inconsistent accuracy
- Interfaces
  - extended and unified input formats
  - callable libraries
- Modeling languages
  - more commercial involvement
- More industrial awareness
  - interrelated with modeling languages

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Other activities

- Software development
  - SeDuMi: software package for conic optimization
  - originally by Jos Sturm
  - in Matlab/C, being reimplemented in Python
  - open source (GPL)
- S-lemma, S-procedure
  - nonconvex quadratic inequalities
- Duality theories in optimization
  - nonconvex
  - nonregular
  - nonexact
- Simulation
  - Canadian Operational Research Society & Visual8
  - Model implemented in Simul8
  - 2007: 1st prize (team member)
  - 2008: results pending (team advisor)

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