Detecting Infeasibility in Conic Optimization: Theory and Practice

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Outline

1. Introduction
   - Importance of infeasibility
2. Theory
   - Classical results
   - Approximate results
   - Interior Point Methods (IPMs)
   - Self-dual embedding
3. Practice
   - Stopping criteria
   - Experiments
   - Results
   - Future plans

Motivation

- Importance of infeasibility
  - No solution.
    - Why? Certificate!
  - What does it mean?
    - Good news?
    - Wrong model? Wrong data?
    - Numerical problems?
    - Bug in the code?
- Practical problems
  - Not known a priori
  - Feasible but impractical solution
    - Missing constraints
  - Weakly infeasible problems

Classical results

- The cone: $\mathcal{K} \subset \mathbb{R}^n$ closed, convex, pointed, solid
- Dual cone: $\mathcal{K}^* = \left\{ s \in \mathbb{R}^n : x^T s \geq 0, \forall x \in \mathcal{K} \right\}$
- Ordering: $x \succeq_{\mathcal{K}} 0 \Leftrightarrow x \in \mathcal{K}$
- Conic optimization problem (CO)

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\succeq_{\mathcal{K}} 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\succeq_{\mathcal{K}^*} 0 \end{aligned}$$

- Weak duality: $c^T x - b^T y = x^T s \geq 0$
- Primal-dual strict feasibility implies strong duality.
- Dual improving direction

$$A^T y \preceq_{\mathcal{K}^*} 0$$
$$b^T y = 1.$$

- Primal is feasible $\Rightarrow$ No dual improving direction
- Primal is infeasible $\Rightarrow$ Almost improving direction

Self-dual cones

- Linear

$$\mathbb{R}^n_\oplus$$

- Second order

$$\mathcal{K}^q = \left\{ x \in \mathbb{R}^n : x_1^2 \geq \|x_{2:n}\|^2 , \, x_1 \geq 0 \right\}$$

- Rotated second order

$$\mathcal{K}^r = \left\{ x \in \mathbb{R}^n : 2x_1x_2 \geq \|x_{3:n}\|^2 , \, x_1, x_2 \geq 0 \right\}$$

- Positive semidefinite

$$\mathbb{S}^n = \left\{ X \in \mathbb{R}^{n \times n} : X \succeq 0 \right\}$$

- And any product of these!

Approximate Farkas Lemma for CO

- Near-solvability

$$\alpha_x = \min \|x\|_\infty \qquad \beta_u = \min \|u\|_1$$
$$Ax = b \qquad A^T y \preceq_{\mathcal{K}^*} u$$
$$x \succeq_{\mathcal{K}} 0 \qquad b^T y = 1$$

Theorem

$\alpha_x \beta_u = 1.$

- Perturbed system:

$$\begin{aligned} \alpha_x^\varepsilon := \min\ & \|x\|_\infty \\ Ax &= b^\varepsilon \\ x &\succeq_{\mathcal{K}} v^\varepsilon \\ \|b - b^\varepsilon\|_\infty &\le \varepsilon \\ \|v^\varepsilon\|_\infty &\le \varepsilon. \end{aligned}$$

- $\alpha_x^\varepsilon \to \alpha_x$ $(\varepsilon \to 0)$
- If the original is feasible then α_x^ε and α_x are realized

IPMs for Conic Optimization

- Optimality conditions

$$\begin{array}{rrcl} Ax & & = & b \\ A^T y & +s & = & c \\ xs & & = & 0 \\ x,\ s & & \succeq & 0, \end{array}$$

- The Newton-system

$$\begin{array}{rrrcl} & A\Delta x & & = & 0 \\ A^T \Delta y & & +\Delta s & = & 0 \\ & \Delta x s & +x\Delta s & = & \mu I - xs \end{array}$$

- Starting point?

IPMs for Conic Optimization

- Perturbed optimality conditions

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ xs &= \mu I \\ x,\ s &\succ 0, \end{aligned}$$

- The Newton-system

$$\begin{aligned} A\Delta x &= 0 \\ A^T \Delta y + \Delta s &= 0 \\ \Delta x s + x\Delta s &= \mu I - xs \end{aligned}$$

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Self-dual embedding
Practice

IPMs for Conic Optimization

- Perturbed optimality conditions

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- Perturbed optimality conditions

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$$\begin{aligned} A\Delta x \quad &= 0 \\ A^T \Delta y \quad + \Delta s &= 0 \\ \Delta x s + x \Delta s &= \mu I - xs \end{aligned}$$

- Starting point?

Self-dual embedding model for CO

- Strictly interior starting point

$$
\begin{array}{rrrrcl}
\multicolumn{6}{c}{\min\ (x_0^T s_0 + 1)\theta} \\
 & Ax & -b\tau & +\bar{b}\theta & = & 0 \\
-A^T y & & +c\tau & -\bar{c}\theta & = & s \\
b^T y & -c^T x & & +\bar{z}\theta & = & \kappa \\
-\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T \tau & & = & -x_0^T s_0 - 1 \\
\multicolumn{6}{c}{x \succeq_{\mathcal{K}} 0,\ \tau \geq 0,\ s \succeq_{\mathcal{K}^*} 0,\ \kappa \geq 0,}
\end{array}
$$

  where
  - $x_0, s_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$, $\tau, \theta, \kappa \in \mathbb{R}$
  - $\bar{b} = b - Ax_0$, $\bar{c} = c - A^T y_0 - s_0$, $\bar{z} = c^T x_0 - b^T y_0 + 1$
- $\tau > 0$, $\theta = 0$: $x/\tau, y/\tau, s/\tau$ are optimal
- $\tau = 0$, $\theta > 0$: infeasibility
- $\tau = 0$, $\theta = 0$: ?

Stopping criteria for the SD model I.

$$\min\ (x_0^T s_0 + 1)\theta$$

$$\begin{array}{rrrrcl} & Ax & -b\tau & +\bar{b}\theta & = & 0 \\ -A^T y & & +c\tau & -\bar{c}\theta & = & s \\ b^T y & -c^T x & & +\bar{z}\theta & = & \kappa \\ -\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T \tau & & = & -x_0^T s_0 - 1 \end{array}$$

$$x \succeq_{\mathcal{K}} 0,\ \tau \geq 0,\ s \succeq_{\mathcal{K}^*} 0,\ \kappa \geq 0,$$

- Stop if
  - optimality:

$$\max\left(\|Ax - b\tau\|,\ \left\|A^T y + s - c\tau\right\|_*,\ c^T x - b^T y\right) < \bar{\varepsilon}\tau$$

  - large primal feasible solutions:

$$b^T y > (\tau \|c\|_* + \theta \|\bar{c}\|_*)\bar{\rho}$$

  - large dual feasible solutions:

$$c^T x > -(\tau \|b\|_* + \theta \left\|\bar{b}\right\|_*)\bar{\rho}$$

- Complexity: $\mathcal{O}\left(\sqrt{n} \ln \frac{\bar{\rho}}{\bar{\varepsilon}}\right)$

Stopping criteria for the SD model II.

- Assume $\tau\kappa \geq (1-\beta)\theta$

$$\begin{array}{rrrrcl} & & & \min\ (x_0^T s_0 + 1)\theta & & \\ & Ax & -b\tau & +\bar{b}\theta & = & 0 \\ -A^T y & & +c\tau & -\bar{c}\theta & = & s \\ b^T y & -c^T x & & +\bar{z}\theta & = & \kappa \\ -\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T\tau & & = & -x_0^T s_0 - 1 \end{array}$$

$$x \succeq_{\mathcal{K}} 0,\ \tau \geq 0,\ s \succeq_{\mathcal{K}^*} 0,\ \kappa \geq 0,$$

- Stop if
  - optimality:

$$\max\left(\|Ax - b\tau\|,\ \left\|A^T y + s - c\tau\right\|_*,\ c^T x - b^T y\right) < \bar{\varepsilon}\tau$$

  - large optimal solutions $({x^*}^T s_0 + {s^*}^T x_0 \geq \rho)$:

$$\tau \leq \frac{1-\beta}{1+\rho}$$

- Complexity: $\mathcal{O}\left(\sqrt{n}\ln\frac{\bar{\rho}}{\bar{\varepsilon}}\right)$

Experiments using SeDuMi

- What is SeDuMi?
  - IPM solver for symmetric cone optimization
  - Self-dual embedding
  - Open source (GPL), Matlab and C
  - Originally by Jos F. Sturm (until 2003)
  - Now by AdvOL (McMaster University)
  - See `http://sedumi.mcmaster.ca`
- Starting point: scaled identity
- Only the first criterion
- Difficulty: few good(!) infeasible problems

A typical infeasible run

```
SeDuMi 1.1 by AdvOL, 2005
eqs m = 10, order n = 31, dim = 901, blocks = 2
it:    b*y     gap    feas  prec   pnorm      tau
 0:         3.56E+1
 1: 7.44E+0 2.40E+1  1.71 3.7E+2  0.0441    1.1137
 2: 1.38E+1 1.16E+1  1.69 1.5E+2  0.1150    1.3545
 3: 1.81E+1 4.51E+0  1.42 5.2E+1  0.1810    1.4779
 4: 4.69E+1 1.47E+0  0.34 3.6E+1  0.5307    0.6770
 5: 3.79E+2 4.73E-1 -1.87 1.0E+2  4.2900    0.0761
 6: 7.04E+2 1.00E-1 -2.15 4.1E+1  7.9618    0.0402
 7: 2.78E+3 3.23E-2 -0.90 5.3E+1 31.4465    0.0313
 8: 1.30E+5 8.78E-4 -1.02 6.7E+1  1.4671E+3 0.0006
 9: 2.31E+5 3.72E-4 -1.12 5.1E+1  2.6118E+3 0.0004
10: 1.14E+6 7.89E-5 -0.98 5.3E+1  1.2870E+4 0.0003
11: 1.14E+8 8.14E-7 -1.00 5.4E+1  1.2888E+6 0
Primal infeasible, dual improving direction found.
```

Observations, conclusions

- It works!
- Useful information
- Special cases, strong bounds
  - SDP, bounded diagonal
  - Natural bounds
- Theoretical properties
  - Polynomial-time stopping criteria
  - Robust, reliable
  - No prior knowledge
  - Numerical accuracy matters

Future work

- Utilize the ${x^*}^T s_0 + {s^*}^T x_0 \geq \rho$ bound
  - Depends on the starting point
  - Primal or dual?
- Weak infeasibility
  - Not implemented anywhere
  - Decision problem
- Derive strong bounds on solutions (preprocessing)
- Statistical methods
  - Similar runs
  - Time series analysis, prediction
  - Algorithm modification
  - No certificate! (uncertainty)

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