Strong duality for optimization over homogeneous cones

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Outline

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   - The Slater condition
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   - Exact dual for the homogeneous case
   - Generalized Schur complement
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The Lagrange-Slater dual

$$\max b^T y \qquad\qquad \min c^T x$$
$$A^T y + s = c \qquad\qquad Ax = b$$
$$s \in \mathcal{K} \qquad\qquad x \in \mathcal{K}^*,$$

Weak duality x, y, s: $c^T x - b^T y \geq 0$ (duality gap)

Strong duality If one problem is strictly feasible

- the other problem is solvable
- zero duality gap at optimality

Without the Slater condition

$$\max\ u_2$$

$$\begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & u_2 & 0 \end{pmatrix} \preceq \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} \min\ \alpha V_{11} \\ V_{22} = 0 \\ V_{11} + 2V_{23} = 1 \\ V \succeq 0 \end{aligned}$$

Primal: $u_2 = 0$, optimum is 0.

Dual: $V_{22} = 0 \rightarrow V_{23} = 0 \rightarrow V_{11} = 1$, optimum is α.

Problem: the feasible set is too small

The regularized problem

- Minimal cone ($\mathcal{K}_{\min}$): spanned by the feasible solutions
- Regularized problem

$$\begin{aligned} \max\ b^T y \\ A^T y + s = c \\ s \in \mathcal{K} \end{aligned} \qquad \begin{aligned} \max\ b^T y \\ A^T y + s = c \\ s \in \mathcal{K}_{\min} \end{aligned} \qquad \begin{aligned} \min\ c^T x \\ Ax = b \\ x \in \mathcal{K}_{\min}^*, \end{aligned}$$

  - equivalent
  - Slater regular

- What is $\mathcal{K}_{\min}$ and $\mathcal{K}_{\min}^*$?
  - construction?
  - structure?
  - complexity?

Homogeneous cones - classical theory

Definition (Vinberg, 1963)

$\mathcal{K}$ is homogeneous if for all $u, v \in \operatorname{int} \mathcal{K}$ there is a linear map M such that $Mu = v$ and $M\mathcal{K} = \mathcal{K}$.

- special polyhedral, Lorentz, semidefinite and much more
- not self-dual (if yes, then symmetric)
- dual is homogeneous
- rank: measure of complexity, $\mathrm{r}(\mathcal{K})$
- $\mathrm{r}(\mathcal{K}) = \mathrm{r}(\mathcal{K}^*)$
- IP complexity: $\mathcal{O}\left(\sqrt{\mathrm{r}(\mathcal{K})}\log(1/\varepsilon)\right)$
- product of elements can be defined

Homogeneous cones - matrix representation

- Generalized matrices:

$$\begin{pmatrix} v_0 & v^T \\ u & u_0 \end{pmatrix}$$

- Multiplication (as usual):

$$\begin{pmatrix} v_0 & v^T \\ u & u_0 \end{pmatrix} \begin{pmatrix} q_0 & q^T \\ p & p_0 \end{pmatrix} = \begin{pmatrix} v_0 q_0 + v^T p & v_0 q^T + p_0 v^T \\ u_0 p + q_0 u & q^T u + u_0 p_0 \end{pmatrix}$$

- Factorization:

$$\begin{pmatrix} v_0 & u^T \\ u & u_0 \end{pmatrix} = \begin{pmatrix} \sqrt{v_0} & 0 \\ \frac{u}{\sqrt{v_0}} & \sqrt{u_0 - \frac{\|u\|^2}{v_0}} \end{pmatrix} \begin{pmatrix} \sqrt{v_0} & \frac{u^T}{\sqrt{v_0}} \\ 0 & \sqrt{u_0 - \frac{\|u\|^2}{v_0}} \end{pmatrix}$$

- Homogeneous cones: 'semidefinite' general matrices
- Rank: 'size' of the matrix

Exact dual for the homogeneous case

- $\mathcal{K}$ is homogeneous

$$\begin{array}{rr} \max\ b^T y & \min\ c^T(x+z^m) \\ A^T y + s = c & A(x+z^m) = b \\ s \in \mathcal{K} & c^T(x^i + z^{i-1}) = 0,\ i = 1,\dots,m \\ & A(x^i + z^{i-1}) = 0,\ i = 1,\dots,m \\ & z^0 = 0 \\ & x^i - z^i z^{i*} \in \mathcal{K},\ i = 1,\dots,m \\ & x \in \mathcal{K}^* \end{array}$$

- zero duality gap
- primal feasible: primal bounded $\Leftrightarrow$ dual feasible
- quadratic constraint?

What is $u - ww^* \in \mathcal{K}$?

- Schur complement for matrices:

$$U - WW^T \succeq 0 \Leftrightarrow \begin{pmatrix} I & W^T \\ W & U \end{pmatrix} \succeq 0$$

- Siegel cone:

$$\mathrm{SC}(\mathcal{K}) = \overline{\{(u, w, t) : t > 0, tu - ww^* \in \mathcal{K}\}}$$

  - if $\mathcal{K}$ is homogeneous $\mathrm{SC}(\mathcal{K})$ is homogeneous
  - every homogeneous cone is a Siegel cone
  - $\mathrm{r}(\mathrm{SC}(\mathcal{K})) = \mathrm{r}(\mathcal{K}) + 1$
  - explicit barrier function can be constructed

Schur complement for second order cones

- Rotated Lorentz cone (homogeneous):

$$\mathbb{L}_r = \left\{ (x_0, x_1, x) \in \mathbb{R}^{n+2} : x_0 \geq 0, x_0 x_1 - \|x\|^2 \geq 0 \right\}$$

- Product:

$$(x_0, x_1, x)(y_0, y_1, y) = (x_0 y_0 + x^T y, x_1 y_1 + x^T y, x_0 y + y_1 x)$$

- $(u_0, u_1, u) - (w_0, w_1, w)^2 \in \mathbb{L}_r \Leftrightarrow$

$$\begin{pmatrix} 1 & \begin{pmatrix} w_0 & w \end{pmatrix} & \begin{pmatrix} w^T & w_1 \end{pmatrix} \\ \begin{pmatrix} w_0 \\ w^T \end{pmatrix} & u_0 & u^T \\ \begin{pmatrix} w \\ w_1 \end{pmatrix} & u & v_0 \end{pmatrix} \in \mathrm{SC}(\mathbb{L}_r)$$

- Everything is linear again!

Conclusions and future research

- Complete duality theory for homogeneous cones
  - without Slater condition
  - explicit dual
- Generalized Schur complement
  - linearization
  - rank increases by 1
- Implementation of homogeneous solvers
  - currently modelled as SDP
  - more efficient modelling with homogeneous cones
- Dual complexity
  - $\mathrm{r}(\mathcal{K}) \geq \mathrm{r}(\mathcal{K}_{\min}) = \mathrm{r}\left(\mathcal{K}_{\min}^{*}\right) \ll \mathrm{r}(\mathcal{K})(m+1)+m$
  - Constructive dual with better complexity?

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