Modelling and optimization with cones

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Symmetric conic optimization in general

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\in \mathcal{K} \end{aligned} \qquad \begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\in \mathcal{K}, \end{aligned}$$

The cone $\mathcal{K}$ can be

Linear: $x \geq 0$

Second-order: $x_0 \geq \|x\|_2$

Rotated second-order: $x_0 x_1 \geq \|x_{2:n}\|$, and $x_0 \geq 0$

Semidefinite: x is (can be assembled into) a symmetric, positive semidefinite matrix

or a product of these.

Semidefinite optimization - notations

The unknown is a matrix:

$$\min \ \mathrm{Tr}(CX) \qquad\qquad \max \ b^T y$$

$$\mathrm{Tr}(A_i X) = b_i, i = 1, \ldots, m \qquad\qquad \sum_{i=1}^{m} A_i y_i + S = C$$

$$X \succeq 0 \qquad\qquad S \succeq 0$$

C, X, S, A_i are $n \times n$ symmetric matrices, $b, y \in \mathbb{R}^m$ vectors

Special structure: A_i, C can be low rank or sparse

Solving conic optimization

- Algorithm: mostly interior point methods
- Iterations
  - SDP: $\mathcal{O}(\sqrt{n})$
  - SOCP: $\mathcal{O}(\sqrt{\#\text{cones}})$
  - practice: $\approx 50 - 100$
- Cost of one iteration
  - SDP: $\mathcal{O}(mn^3 + m^2n^2 + m^3)$
  - SOCP: $\mathcal{O}(m^3 + ...)$
  - much less for sparse data
- Problem sizes:
  - SDP: $m \leq 10000$, $n \leq 5000$
  - SOCP $m \leq 10000$, $k \leq 10000$
  - more with sparse data
- High accuracy

QP as SOCP

Quadratic optimization

$$\begin{aligned} \min\ & x^T Q x + c^T x \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Cholesky factorization: $LL^T = Q$

SOCP formulation:

$$\begin{aligned} \min\ & u_0 \\ Lx - u &= \frac{1}{2} L^{-1} c \\ Ax &= b \\ x &\geq 0 \\ u_0 &\geq \|u\| \end{aligned}$$

Quadratic/linear fractions SOCP

$$\sum_{i=1}^{r} \frac{\|A_i x + b_i\|^2}{a_i^T x + \beta_i} \leq t$$

can be represented as

$$\begin{aligned} \sum_{i=1}^{r} u_i &\leq t \\ w_i^T w_i &\leq u_i v_i,\ i = 1, \ldots, r \\ w_i &= A_i x + b_i,\ i = 1, \ldots, r \\ v_i &= a_i^T x + \beta_i \geq 0,\ i = 1, \ldots, r \end{aligned}$$

Inequalities with rational powers SOCP

$$x_1^{-5/6} x_2^{-1/3} x_3^{-1/2} \le t$$
$$x_1, x_2, x_3 \ge 0$$

Integer powers:

$$1 \le x_1^5 x_2^2 x_3^3 t^6$$
$$x_1, x_2, x_3 \ge 0$$

Equivalent formulation (rotated SOCP):

$$\begin{aligned}
w_1^2 &\le x_1 x_3 & & (1 \le x_1^4 x_2^2 x_3^2 t^6 w_1^2)\\
w_2^2 &\le x_3 w_1 & & (1 \le x_1^4 x_2^2 t^6 w_2^4)\\
w_3^2 &\le x_2 t & & (1 \le x_1^4 t^4 w_3^4 w_2^4)\\
w_4^2 &\le w_2 w_3 & & (1 \le x_1^4 t^4 w_4^8)\\
w_5^2 &\le x_1 t & & (1 \le w_4^8 w_5^8)\\
1 &\le w_4 w_5 & &
\end{aligned}$$

Not unique!

Robust optimization

Linear optimization with uncertain data:

$$\min\ c^T x$$
$$a_j^T x - b_j \geq 0, \forall j = 1, \dots, m,$$

where

$$\left\{ \begin{pmatrix} a_j \\ -b_j \end{pmatrix} = \begin{pmatrix} a_j^0 \\ -b_j^0 \end{pmatrix} + Pu : u \in \mathbb{R}^m,\ \|u\| \leq 1 \right\},$$

Robust LP:

$$\left(\begin{pmatrix} a_j^0 \\ -b_j^0 \end{pmatrix} + Pu \right)^T \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0,\ \forall u : \|u\| \leq 1,$$

SOCP form:

$$\left(a_j^0\right)^T x - b_j^0 \geq \left\| P^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|.$$

Eigenvalue optimization SDP

Problem

Given matrices $A_1, \ldots, A_m$ find a nonnegative linear combination whose smallest eigenvalue is maximal.

SDP formulation

$$\max \lambda$$

$$\sum_{i=1}^{m} A_i y_i - \lambda I \succeq 0$$

$$y_i \geq 0, \; i = 1, \ldots, m$$

Needed for stability analysis

Relaxation of binary variables

SDP

Binary constraints: $x_i \in \{0,1\}$

LP relaxation: $x_i \in [0,1]$

Equivalent form:

$$z_i = 2x_i - 1$$
$$z_i^2 = 1 (\Leftrightarrow z_i = \pm 1)$$

Matrix form:

$$Z \succeq 0$$
$$\mathrm{diag}(Z) = 1$$
$$\mathrm{rank}(Z) = 1 (\Leftrightarrow Z = zz^T)$$

Graph partitioning SDP

Given $2m$ points with weighted edges between them, divide them into 2 equal partitions such that the total weight of edges between the two partitions is minimized.

A: incidence matrix, A_{kl} is the weight of the kl edge

$y_{ij} = 1$: point i is in partition j

y_j: the indicator vector of partition j

$y_j^T A y_j$: $2\times$ the weight of edges with both ends in partition j

$\mathrm{Tr}(Y^T A Y)$: twice the total weight of uncut edges

$e^T A e$: twice total weight of edges

$$\min\ e^T A e - \mathrm{Tr}(Y^T A Y)$$

Y is a partition matrix

SDP relaxation ($X = YY^T$) $\Rightarrow$

$$\begin{aligned} \min\ & e^T A e - \mathrm{Tr}(AX) \\ & \mathrm{diag}(X) = 1 \\ & Xe = m \\ & X \geq 0 \\ & X \succeq 0 \end{aligned}$$

Stability analysis SDP

Theorem (Lyapunov stability)

Trajectories of $\dot{x} = Ax$ converge to 0 if and only if there is a positive definite P such that $A^T P + PA$ is negative definite.

Theorem (Circle criterion)

Linear state-feedback system:

$$\dot{x} = Ax + Bw, \quad x(0) = x_0$$
$$v = Cx$$
$$\sigma(v, w) = (\beta v - w)^T (w - \alpha v) \geq 0, \quad \text{(sector constraint, } \alpha < \beta\text{)}$$

Equivalent condition for quadratic stability:

$$P \succ 0$$
$$\begin{pmatrix} A^T P + PA - 2\beta\alpha C^T C & PB + (\beta + \alpha) C^T \\ B^T P + (\beta + \alpha) C & -2 \end{pmatrix} \prec 0$$

Structural design I — SDP

M: degrees of freedom, (number of joints)

$f \in \mathbb{R}^M$: loads on the joints

$\mathcal{F}$: set of loading scenarios

$w \in \mathbb{R}^M$: displacements of the joints

N: number of bars

$t \in \mathbb{R}^N$: volume of bars (←design variables)

$Q(t) = \sum_{i=1}^N b_i b_i^T t_i$: energy matrix of the truss

$\frac{1}{2} w^T Q(t) w$: potential energy for displacement w

$f^T w - \frac{1}{2} w^T Q w$: compliance, maximized by w

Goal: for a given set of loads design a truss with minimum worst case compliance

Constraints: volumes of bars (individual and total), obstacles, maximum displacements

Structural design II

SDP

Design problem (multiload):

$$\min_t \left\{ \sup_{f,w} f^T w - \frac{1}{2} w^T Q(t) w \right\}$$

$$t_i \in \left[\underline{t_i}, \overline{t_i}\right] \text{ (lower/upper bound on volumes)}$$

$$\sum_{i=1}^{N} t_i \leq v \text{ (total volume)}$$

Semidefinite reformulation:

$$\min_t \ \tau$$

$$\begin{pmatrix} \tau & f_i^T \\ f_i & Q \end{pmatrix} \succeq 0, \ i = 1, \dots, k$$

$$t_i \in \left[\underline{t_i}, \overline{t_i}\right] \text{ (lower/upper bound on volumes)}$$

$$\sum_{i=1}^{N} t_i \leq v \text{ (total volume)}$$

Polynomial optimization I SDP

Theorem

If $p(x) : \mathbb{R} \to \mathbb{R}$ *is a univariate polynomial*
$p(x) \geq 0, \forall x \Leftrightarrow p(x)$ *is a sum-of-squares (SOS)*

Example

$p(x) = x^6 - 5x^4 + 6x^3 + 8x^2 - 14x + 5 =$
$(x^2 + x - 1)^2 + (x^3 - 3x + 2)^2$ is SOS

Example

Converse is not true:
$p(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \geq 0$, but NOT SOS

$$\min\ p(x) \quad \Leftrightarrow \quad \begin{array}{c}\max t \\ p(x) - t \geq 0,\ \forall x\end{array} \quad \Leftrightarrow \quad \begin{array}{c}\max t \\ p(x) - t \text{ is SOS}\end{array}$$

Polynomial optimization II — SDP

$q = (1, x, x^2, \ldots, x^n)$

Square:

$$p(x) = \left(\sum_{i=0}^{n} u_i x^i\right)^2 = \left(\sum_{i=0}^{n} u_i q_i\right)^2 = (u^T q)^2 = q^T (uu^T) q$$

SOS: $q^T U q$, where U is PSD

$$\begin{aligned} u_{44} &= 1 \\ u_{34} + u_{43} &= 0 \\ u_{24} + u_{33} + u_{42} &= -5 \\ u_{14} + u_{23} + u_{32} + u_{41} &= 6 \\ u_{13} + u_{22} + u_{31} &= 8 \\ u_{21} + u_{12} &= -14 \\ u_{11} &= 5 \\ U &\succeq 0 \end{aligned}$$

$$U = \begin{pmatrix} 5 & -7 & -1 & 2 \\ -7 & 10 & 1 & -3 \\ -1 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{pmatrix}$$

$$u^{(1)} = \begin{pmatrix} -1 & 1 & 1 & 0 \end{pmatrix}$$

$$u^{(2)} = \begin{pmatrix} 2 & -3 & 0 & 1 \end{pmatrix}$$

Extensions to more variables, interval constraints

Software: (Gloptipoly, SOSTools), Yalmip

Software

- Solvers - mostly open source, last 10 years

  SOCP: MOSEK, SeDuMi, LOQO, SDPT3, PENNON

  SDP: CSDP, SDPA, SDPA, SeDuMi, SDPT3, PENSDP
- Modelling languages
  - No commercial support for SDP
  - GAMS: only SOCP
  - Yalmip: Matlab based, very powerful
  - CVX: Matlab, more academic

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