2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities
**Affine set**

*line* through $x_1$, $x_2$: all points

\[ x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbb{R}) \]

**affine set**: contains the line through any two distinct points in the set

**example**: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)
Convex set

**line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)
Convex combination and convex hull

**convex combination** of $x_1, \ldots, x_k$: any point $x$ of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

**convex hull** $\text{conv } S$: set of all convex combinations of points in $S$
Convex cone

conic (nonnegative) combination of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

convex cone: set that contains all conic combinations of points in the set
Hyperplanes and halfspaces

**hyperplane**: set of the form \( \{ x \mid a^T x = b \} \) \( (a \neq 0) \)

- \( a \) is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

**halfspace**: set of the form \( \{ x \mid a^T x \leq b \} \) \( (a \neq 0) \)
Euclidean balls and ellipsoids

(Euclidean) ball with center $x_c$ and radius $r$:

$$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \leq r \} = \{ x_c + ru \mid \|u\|_2 \leq 1 \}$$

ellipsoid: set of the form

$$\{ x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1 \}$$

with $P \in \mathbb{S}^n_{++}$ (i.e., $P$ symmetric positive definite)

other representation: $\{ x_c + Au \mid \|u\|_2 \leq 1 \}$ with $A$ square and nonsingular
Norm balls and norm cones

**norm:** a function \( \| \cdot \| \) that satisfies

- \( \|x\| \geq 0; \|x\| = 0 \) if and only if \( x = 0 \)
- \( \|tx\| = |t| \|x\| \) for \( t \in \mathbb{R} \)
- \( \|x + y\| \leq \|x\| + \|y\| \)

**notation:** \( \| \cdot \| \) is general (unspecified) norm; \( \| \cdot \|_{\text{symb}} \) is particular norm

**norm ball** with center \( x_c \) and radius \( r \): \( \{ x \mid \|x - x_c\| \leq r \} \)

**norm cone:** \( \{ (x, t) \mid \|x\| \leq t \} \)

Euclidean norm cone is called second-order cone

**norm balls and cones are convex**
Polyhedra

solution set of finitely many linear inequalities and equalities

\[ Ax \preceq b, \quad Cx = d \]

\((A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}, \preceq \text{ is componentwise inequality})\)

polyhedron is intersection of finite number of halfspaces and hyperplanes
Positive semidefinite cone

notation:
- \( S^n \) is set of symmetric \( n \times n \) matrices
- \( S^n_+ = \{ X \in S^n \mid X \succeq 0 \} \): positive semidefinite \( n \times n \) matrices
  \[ X \in S^n_+ \iff z^T X z \geq 0 \text{ for all } z \]

\( S^n_+ \) is a convex cone
- \( S^n_{++} = \{ X \in S^n \mid X \succ 0 \} \): positive definite \( n \times n \) matrices

example: \( \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+ \)
Operations that preserve convexity

practical methods for establishing convexity of a set \( C \)

1. apply definition

\[
x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C
\]

2. show that \( C \) is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . . ) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions
Intersection

the intersection of (any number of) convex sets is convex

example:

\[ S = \{ x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3 \} \]

where \( p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt \)

for \( m = 2 \):
Affine function

suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is affine (\( f(x) = Ax + b \) with \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \))

- the image of a convex set under \( f \) is convex

\[ S \subseteq \mathbb{R}^n \text{ convex} \implies f(S) = \{ f(x) \mid x \in S \} \text{ convex} \]

- the inverse image \( f^{-1}(C) \) of a convex set under \( f \) is convex

\[ C \subseteq \mathbb{R}^m \text{ convex} \implies f^{-1}(C) = \{ x \in \mathbb{R}^n \mid f(x) \in C \} \text{ convex} \]

examples

- scaling, translation, projection
- solution set of linear matrix inequality \( \{ x \mid x_1A_1 + \cdots + x_mA_m \preceq B \} \) (with \( A_i, B \in \mathbb{S}^p \))
- hyperbolic cone \( \{ x \mid x^TPx \leq (c^Tx)^2, c^Tx \geq 0 \} \) (with \( P \in \mathbb{S}_+^n \))
**Perspective and linear-fractional function**

**Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x, t) = \frac{x}{t}, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

Images and inverse images of convex sets under perspective are convex.

**Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(x) = \frac{Ax + b}{c^Tx + d}, \quad \text{dom } f = \{x \mid c^Tx + d > 0\}$$

Images and inverse images of convex sets under linear-fractional functions are convex.
example of a linear-fractional function

\[ f(x) = \frac{1}{x_1 + x_2 + 1} \]
Generalized inequalities

A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

- $K$ is closed (contains its boundary)
- $K$ is solid (has nonempty interior)
- $K$ is pointed (contains no line)

**Examples**

- Nonnegative orthant $K = \mathbb{R}^+_n = \{ x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n \}$
- Positive semidefinite cone $K = \mathbb{S}^+_n$
- Nonnegative polynomials on $[0, 1]$:

$$K = \{ x \in \mathbb{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1] \}$$
**generalized inequality** defined by a proper cone $K$:

$$x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \text{int } K$$

**examples**

- componentwise inequality ($K = \mathbb{R}^n_+$)
  $$x \preceq_{\mathbb{R}^n_+} y \iff x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbb{S}^n_+$)
  $$X \preceq_{\mathbb{S}^n_+} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

**properties:** many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$
Minimum and minimal elements

$\leq_K$ is not in general a linear ordering: we can have $x \not\leq_K y$ and $y \not\leq_K x$

$x \in S$ is the minimum element of $S$ with respect to $\leq_K$ if

$$y \in S \implies x \leq_K y$$

$x \in S$ is a minimal element of $S$ with respect to $\leq_K$ if

$$y \in S, \quad y \leq_K x \implies y = x$$

example ($K = \mathbb{R}^2_+$)

$x_1$ is the minimum element of $S_1$

$x_2$ is a minimal element of $S_2$
The Separating hyperplane theorem states that if $C$ and $D$ are disjoint convex sets, then there exists $a \neq 0$, $b$ such that:

\[
\begin{align*}
    a^T x &\leq b \quad \text{for } x \in C, \\
    a^T x &\geq b \quad \text{for } x \in D
\end{align*}
\]

Thus, the hyperplane $\{x \mid a^T x = b\}$ separates $C$ and $D$.

Strict separation requires additional assumptions (e.g., $C$ is closed, $D$ is a singleton).
Supporting hyperplane theorem

supporting hyperplane to set $C$ at boundary point $x_0$:

$$\left\{ x \mid a^T x = a^T x_0 \right\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

**supporting hyperplane theorem**: if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$
Dual cones and generalized inequalities

dual cone of a cone $K$:

$$K^* = \{ y \mid y^T x \geq 0 \text{ for all } x \in K \}$$

examples

- $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
- $K = S^n_+$: $K^* = S^n_+$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$

first three examples are self-dual cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$
Minimum and minimal elements via dual inequalities

**minimum element** w.r.t. $\preceq_K$

$x$ is minimum element of $S$ iff for all $\lambda \succ_K^* 0$, $x$ is the unique minimizer of $\lambda^T z$ over $S$

**minimal element** w.r.t. $\preceq_K$

- if $x$ minimizes $\lambda^T z$ over $S$ for some $\lambda \succ_K^* 0$, then $x$ is minimal

- if $x$ is a minimal element of a *convex* set $S$, then there exists a nonzero $\lambda \succeq_K^* 0$ such that $x$ minimizes $\lambda^T z$ over $S$
optimal production frontier

- Different production methods use different amounts of resources $x \in \mathbb{R}^n$
- Production set $P$: resource vectors $x$ for all possible production methods
- Efficient (Pareto optimal) methods correspond to resource vectors $x$ that are minimal w.r.t. $\mathbb{R}_+^n$

**Example** ($n = 2$)

$x_1, x_2, x_3$ are efficient; $x_4, x_5$ are not