CES 735 Assignment #1 Due date: Friday, February 1, 4:00pm

Dr. Imre Pólik (50 marks)

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1. (5 marks) What is the distance between two parallel hyperplanes
$$\left\{x \in \mathbb{R}^n : a^T x = b_1\right\} \text{ and } \left\{x \in \mathbb{R}^n : a^T x = b_2\right\}? \tag{1}$$

2. (10 marks) Which of the following sets are polyhedra? If possible express them in the standard form presented in class:
   (a) $S = \{y_1 a_1 + y_2 a_2 : -1 \le y_1 \le 1,\ -1 \le y_2 \le 1\}$ where $a_1, a_2 \in \mathbb{R}^n$;
   (b) $S = \left\{x \in \mathbb{R}^n : x \ge 0,\ e^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\right\}$, where $a_1, \ldots, a_n \in \mathbb{R}$, $b_1, b_2 \in \mathbb{R}$ and $e = (1, \ldots, 1)$;
   (c) $S = \left\{x \in \mathbb{R}^n : x \ge 0,\ x^T y \le 1 \text{ for all } y \text{ with } \|y\|_2 = 1\right\}$;
   (d) $S = \left\{x \in \mathbb{R}^n : x \ge 0,\ x^T y \le 1 \text{ for all } y \text{ with } \|y\|_1 = 1\right\}$.

3. (5 marks) Express the convex set $\left\{x \in \mathbb{R}^2_+ : x_1 x_2 \ge 1\right\}$ as an intersection of halfspaces.

4. (6 marks) Is the set of real, positive semidefinite, not necessarily symmetric matrices a proper cone? Verify all the requirements, or give a counterexample if one is not satisfied.

5. (10 marks) For each of the following functions determine whether it is convex, quasiconvex, concave or quasiconcave:
   (a) $f(x) = e^x - 1$, $x \in \mathbb{R}_+$;
   (b) $f(x) = \operatorname{Tr}(X^{-1})$, where X is symmetric and positive definite;
   (c) $f(x) = \frac{a^T x + b}{c^T x + d}$, where $c^T x + d > 0$;
   (d) $f(x, t) = -\log(t^2 - x^T x)$, on $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t > \|x\|\}$;
   (e) $f(x) = \sum_{i=1}^n x_i \log x_i - (\sum_{i=1}^n x_i) \log(\sum_{i=1}^n x_i)$ where $x_i > 0$ for $i = 1, \ldots, n$;

6. (7 marks) Recall the illumination problem from lecture 1.
   (a) Is the objective function $(\max_i |\log I_i - \log I_{\text{des}}|)$ convex in the variables $p_1, \ldots, p_m$?
   (b) Show that the constraint $0 \le p_i \le p_{\max}$ together with the constraint that no more

than half of the total power is in any k lamps ($k \leq m$) is convex.

(c) Show that the constraint $0 \leq p_i \leq p_{\max}$ together with the constraint that no more than half of the lamps are on is (in general) not convex.

7. (7 marks) Show that the distance ratio function, i.e.,

$$f(x) = \frac{\|x - a\|}{\|x - b\|} \tag{2}$$

is quasiconvex on the halfspace $\{x : \|x - a\| \leq \|x - b\|\}$. What are the sublevel sets?