1. (20 points) Consider the following matrix:

\[ A \in \mathbb{R}^{n^2 \times n^2} \]

\[ A \text{vec}(X) = \text{vec}(UXU), \]

where \( U \) and \( X \) are \( n \times n \) symmetric matrices, and \( \text{vec}(X) \) is the vectorization of \( X \), i.e., it contains the columns of \( X \) stacked together in one long column vector. Matrix \( A \) maps an \( n \times n \) matrix to another one.

(a) Form \( A \) explicitly in terms of \( U \). What is the structure of this matrix? (Hint: \( \otimes \))

(b) Analyze the applicability of direct methods to find the eigenvalues of \( A \). Do they work here?

(c) Now consider iterative methods. Choose a suitable algorithm to find the eigenvalues of \( A \). Try it for some simple cases when you know the eigenvalues of \( U \). What do you observe?

(d) After all, what is the best way to find the eigenvalues of this matrix? Can you get them directly from \( U \)?

2. (20 points) Implement the bisection algorithm to find the eigenvalues of a matrix in an interval \([\alpha, \beta]\). You can use the pseudocode on page 229 in Demmel’s book. Experiment with different strategies to choose the bisecting point: midpoint, golden section, random, two bisecting points at the thirds, etc.

**Bonus:** (20 points) Find an efficient algorithm to compute the QR decomposition of a matrix \( A = R + uv^T \), where \( R \) is upper triangular, and \( u \) and \( v \) are column vectors. (Hint: Use Givens rotations. The complexity of your algorithm should be \( O(n^2) \) instead of \( O(n^3) \).) [From James W. Demmel. Applied Numerical Linear Algebra. SIAM, Philadelphia, 1997.]