Rounding solutions in SOCP

Imre Pólik, Julio C. Góez
SAS Institute
Cary, NC, USA

ICCOPT 2013, Lisbon

Outline

Problem definition

SOCP, primal form

$$
\begin{aligned}
\min \ & \sum_{i=1}^{K} (c_0^i, c^i)^T (x_0^i, x^i) \\
& \sum_{i=1}^{K} (A_0^i, A^i)(x_0^i, x^i) = b \\
& \qquad\qquad (x_0^i, x^i) \in \mathcal{K}_i, \ i = 1, \ldots, K,
\end{aligned}
$$

Problem definition

SOCP, primal form

$$\min \sum_{i=1}^{K} \mathbf{c^i}^T \mathbf{x^i}$$

$$\sum_{i=1}^{K} \mathbf{A^i x^i} = b$$

$$\mathbf{x^i} \in \mathcal{K}_i,\ i = 1, \ldots, K.$$

Problem definition

SOCP, primal form

$$\min \sum_{i=1}^{K} \mathbf{c^i}^T \mathbf{x^i}$$

$$\sum_{i=1}^{K} \mathbf{A^i x^i} = b$$

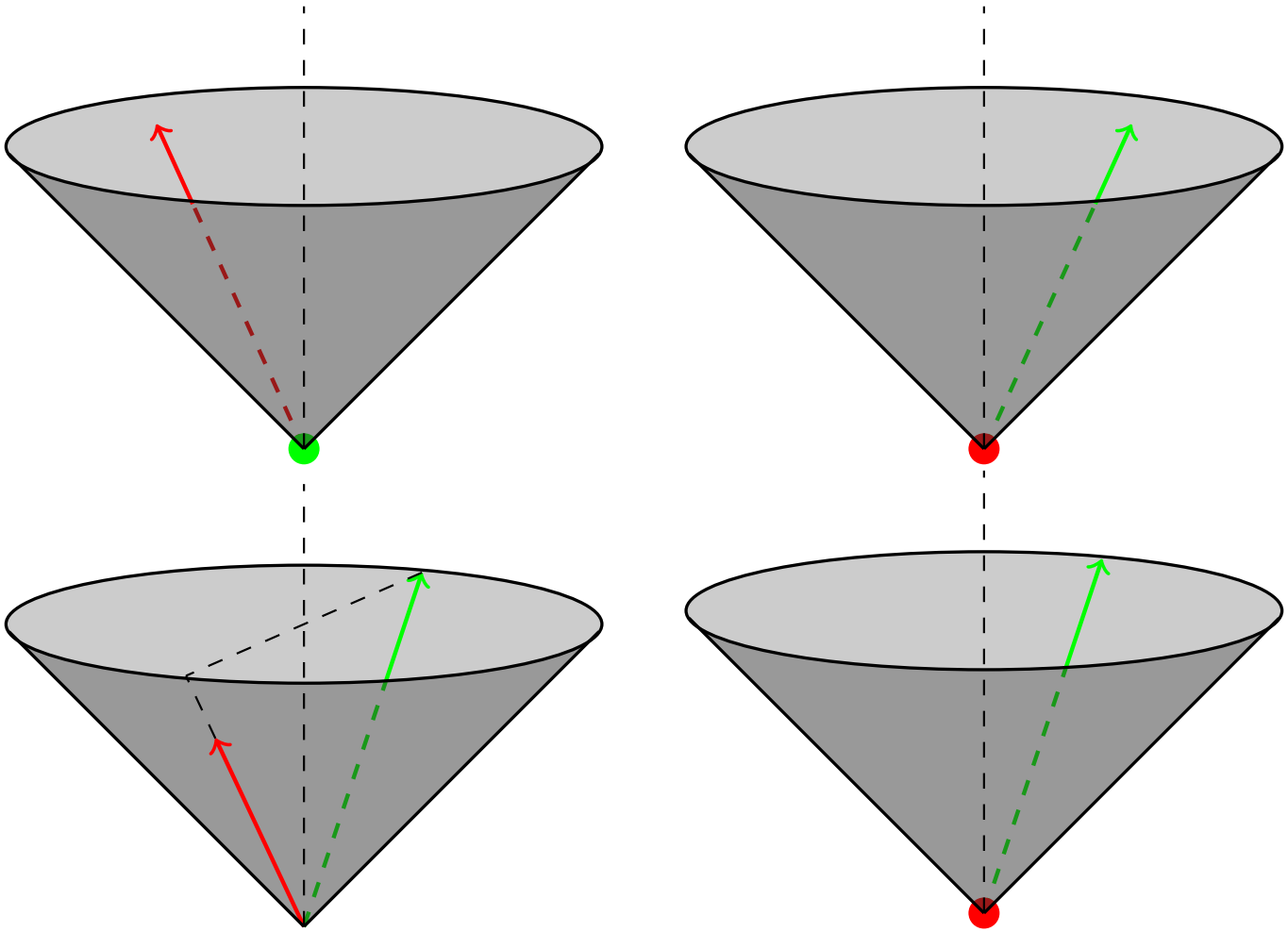
$$\mathbf{x^i} \in \mathcal{K}_i,\ i = 1, \ldots, K.$$

Algorithms

Solved with IPM

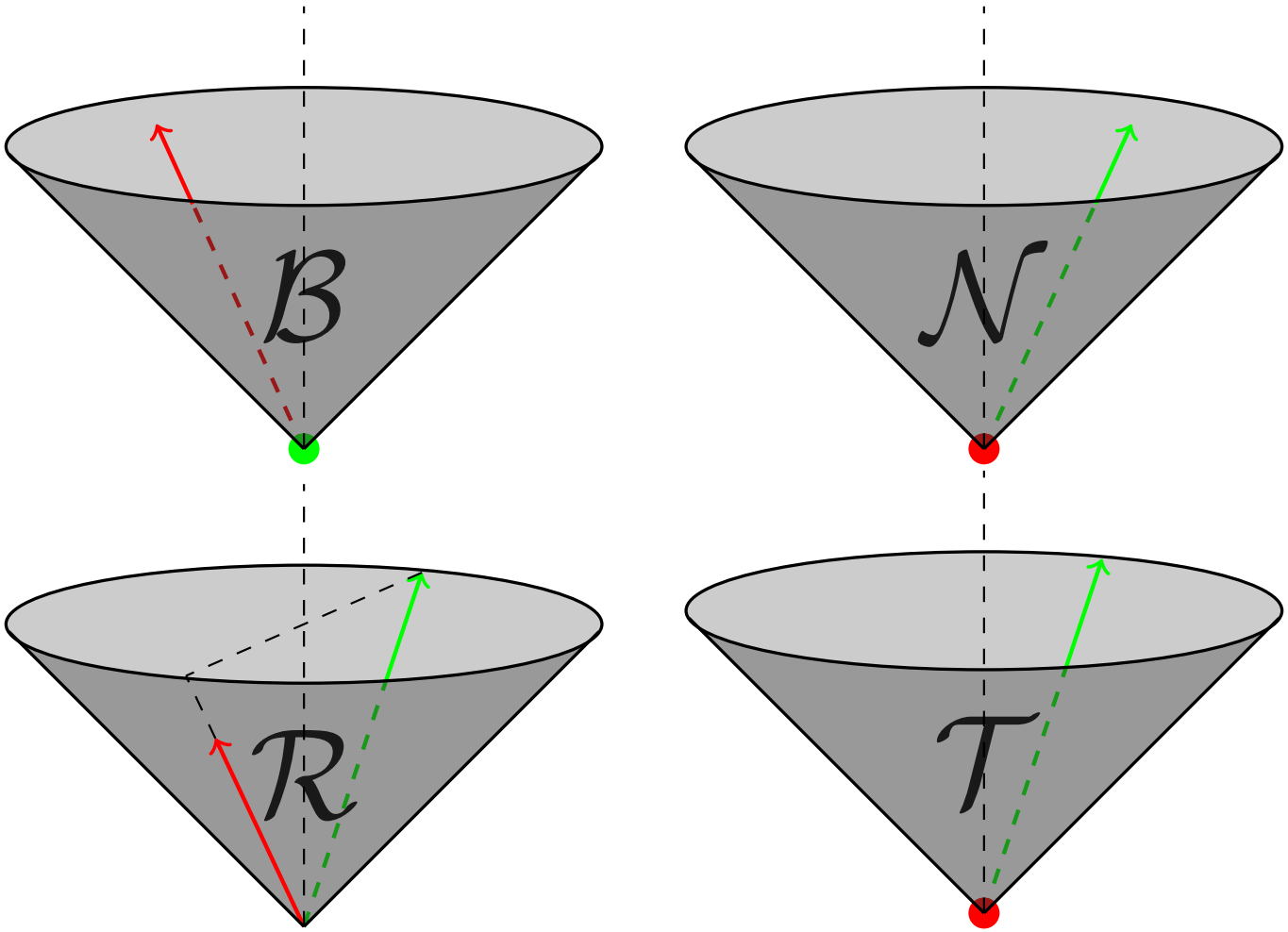
Optimal solution is dense (high rank)

Complementarity and optimal partitions



Terlaky, Wang: *On identification of the optimal partition of SOCPs*

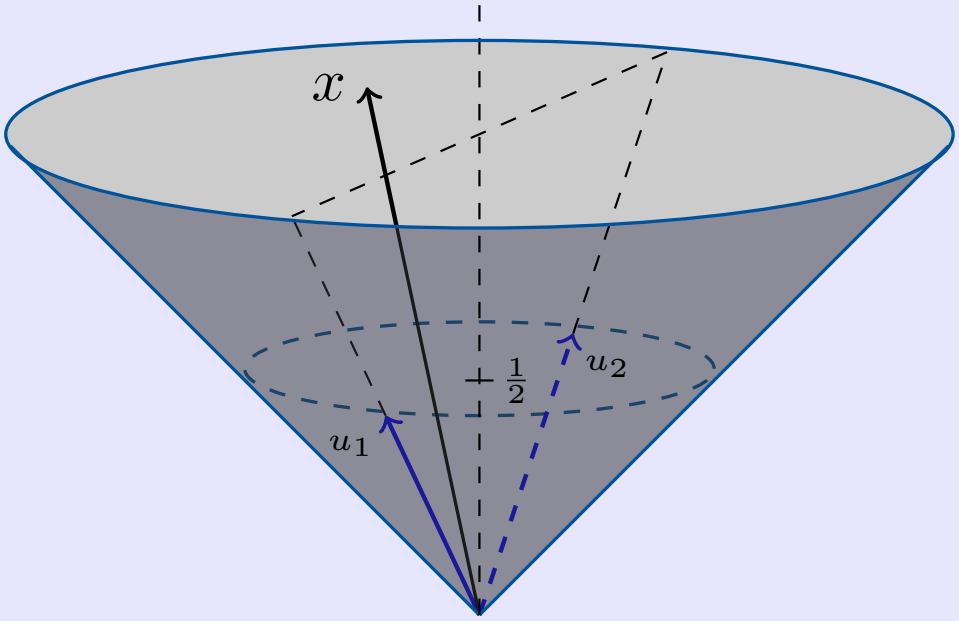
Complementarity and optimal partitions



Terlaky, Wang: *On identification of the optimal partition of SOCPs*

Jordan frames

Rank-1 decomposition for SOCP



$$x = \lambda_1 u_1 + \lambda_2 u_2$$

$u_1, u_2 \in \mathcal{K}$, $u_1^T u_2 = 0$,
$\|u_1\| = \|u_2\| = 1/\sqrt{2}$

λ_1, λ_2: Jordan values
u_1, u_2: Jordan frame

Easy to compute

$$\mathbf{x} = (x_0, x) = \underbrace{(x_0 + \|x\|)}_{\lambda_1} \underbrace{\left(\frac{1}{2}; \frac{x}{2\|x\|}\right)}_{u_1} + \underbrace{(x_0 - \|x\|)}_{\lambda_2} \underbrace{\left(\frac{1}{2}; \frac{-x}{2\|x\|}\right)}_{u_2}$$

Optimal rounding LP

Algorithm

1. Take an optimal IPM solution
2. Fix its Jordan frame
3. Optimize the Jordan values
4. Form a new solution with the original Jordan frame and the optimized Jordan values

Properties

- Two nonnegative variables per cone
- Rounded solution is still optimal
- The dual solution is dual feasible for the original SOCP
- At most m Jordan values are nonzero (sparse solution)

Optimal rounding LP

Decomposition

Optimal solution: $\mathbf{x} = \left(\mathbf{x^1}, \ldots, \mathbf{x^K}\right)$.

$$\mathbf{x^i} = (x_0^i, x^i) = \bar{\lambda}_+^i \left(\frac{1}{2}; u^i\right) + \bar{\lambda}_-^i \left(\frac{1}{2}; -u^i\right),\ i = 1, \ldots, K.$$

where $\left\|u^i\right\| = 1/2$ and $\bar{\lambda}_+^i, \bar{\lambda}_-^i \geq 0$.

Optimal rounding LP

Decomposition

Optimal solution: $\mathbf{x} = \left(\mathbf{x^1}, \ldots, \mathbf{x^K}\right)$.

$$\mathbf{x^i} = (x_0^i, x^i) = \bar{\lambda}_+^i \left(\frac{1}{2}; u^i\right) + \bar{\lambda}_-^i \left(\frac{1}{2}; -u^i\right), \; i = 1, \ldots, K.$$

where $\left\|u^i\right\| = 1/2$ and $\bar{\lambda}_+^i, \bar{\lambda}_-^i \geq 0$.

Rounding

$$\begin{aligned}
\min \; & \sum_{i=1}^{K} \lambda_+^i (c_0^i, c^i)^T \left(\frac{1}{2}; u^i\right) + \sum_{i=1}^{K} \lambda_-^i (c_0^i, c^i)^T \left(\frac{1}{2}; -u^i\right) \\
& \sum_{i=1}^{K} \lambda_+^i (A_0^i, A^i) \left(\frac{1}{2}; u^i\right) + \sum_{i=1}^{K} \lambda_-^i (A_0^i, A^i) \left(\frac{1}{2}; -u^i\right) = b \\
& \lambda_+^i, \lambda_-^i \geq 0, \; i = 1, \ldots, K
\end{aligned}$$

Example

$$\min 0$$
$$1x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$-1x_1 + 2x_2 - 3x_3 + 4x_4 = 0$$
$$x_1 \geq 0$$
$$x_2 \geq \sqrt{x_3^2 + x_4^2}$$

IPM

Optimal solution:
$x = (2.98, 7.57, -3.12, -5.38)$
Jordan values: $2.98, 0.95, 9.75$
Rank: 3
Duality: (**B**, **B**)

Rounding

Optimal solution:
$x = (0.08, 1.68, -0.84, -1.45)$
Jordan values: $0.08, 0, 2.38$
Rank: 2
Duality: (**B**, **T**)

Example

$$\min 0$$
$$1x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$-1x_1 + 2x_2 - 3x_3 + 4x_4 = 0$$
$$x_1 \geq 0$$
$$x_2 \geq \sqrt{x_3^2 + x_4^2}$$

IPM

Optimal solution:
$x = (2.98, 7.57, -3.12, -5.38)$
Jordan values: $2.98, 0.95, 9.75$
Rank: 3
Duality: (**B**, **B**)

Rounding

Optimal solution:
$x = (0.08, 1.68, -0.84, -1.45)$
Jordan values: $0.08, 0, 2.38$
Rank: 2
Duality: (**B**, **T**)

How good does the optimal solution need to be?

Rounding intermediate solutions

Algorithm

1. Take an intermediate IPM solution
2. Fix its Jordan frame
3. Optimize the Jordan values
4. Form a new solution with the optimized Jordan values
5. Stop IPM if rounded solution is good

Issues

- Intermediate solutions are not feasible
  - » they still have a Jordan frame, take primal, dual or both
- Primal and dual Jordan frames are different
  - » lack of complementarity
  - » the rounded solution may not be feasible
  - » infeasibility measures solution quality
- Rounding problem may be infeasible

How good is it?

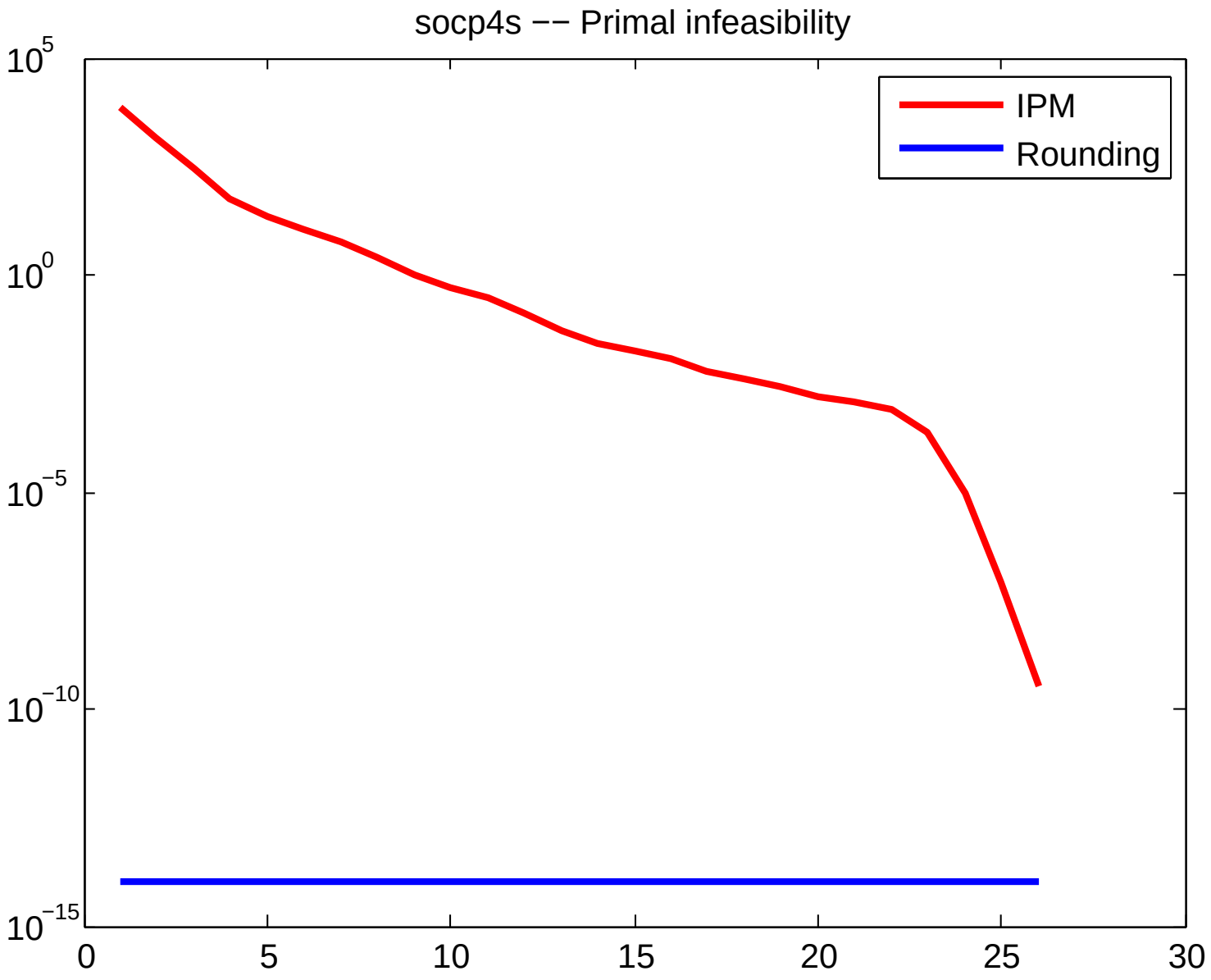
Computational setup

- 25 instances
  - » FIR filter
  - » scheduling
  - » robust radiation therapy treatment planning
- IPM solves with SeDuMi 1.3 in Matlab R2011a (64bit)
- Primal rounding with CPLEX 12.5 (default on 15 threads)
- Experiments run by Julio Góez at Lehigh University

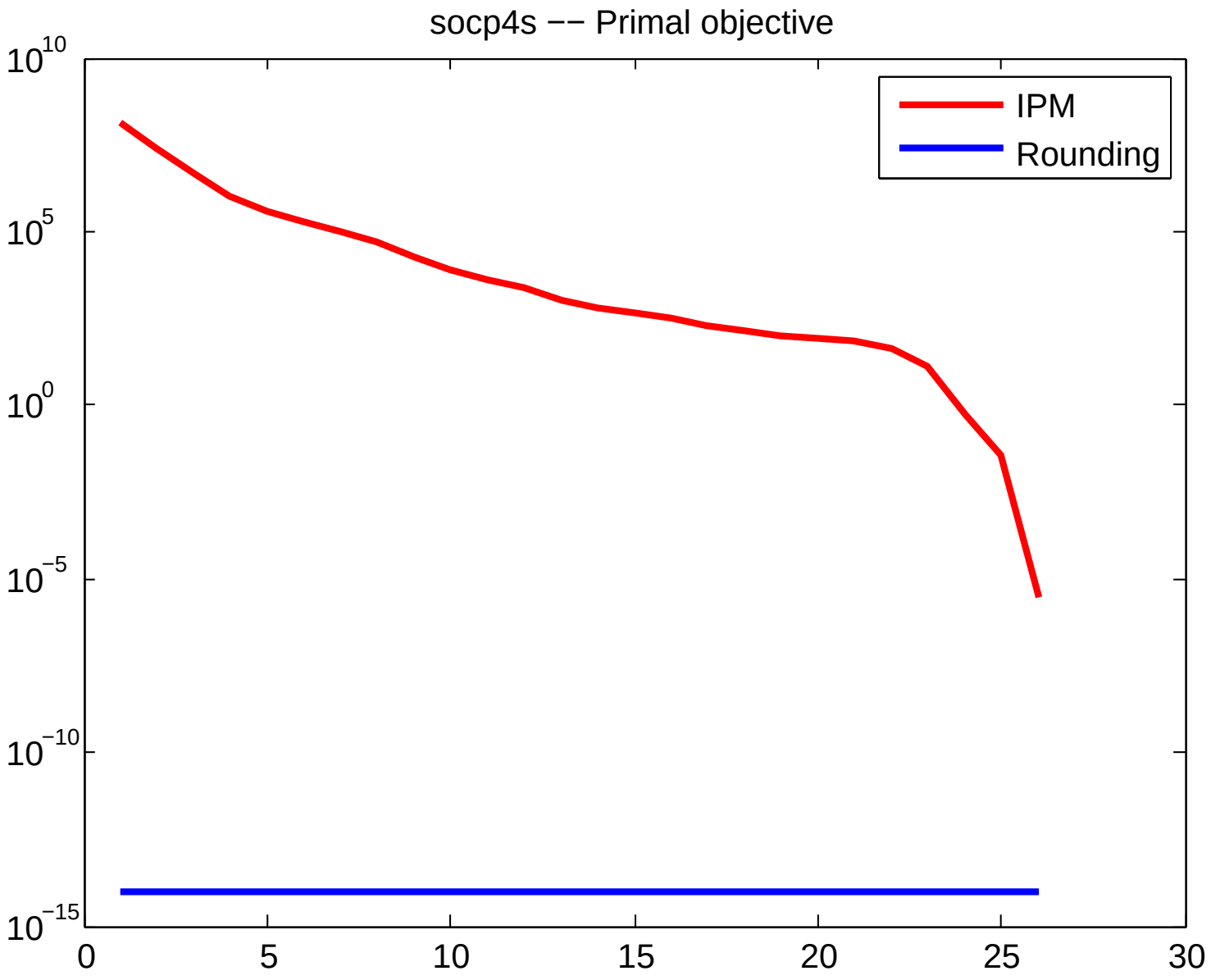
General findings

Some of the LPs are very hard
Problems behave differently

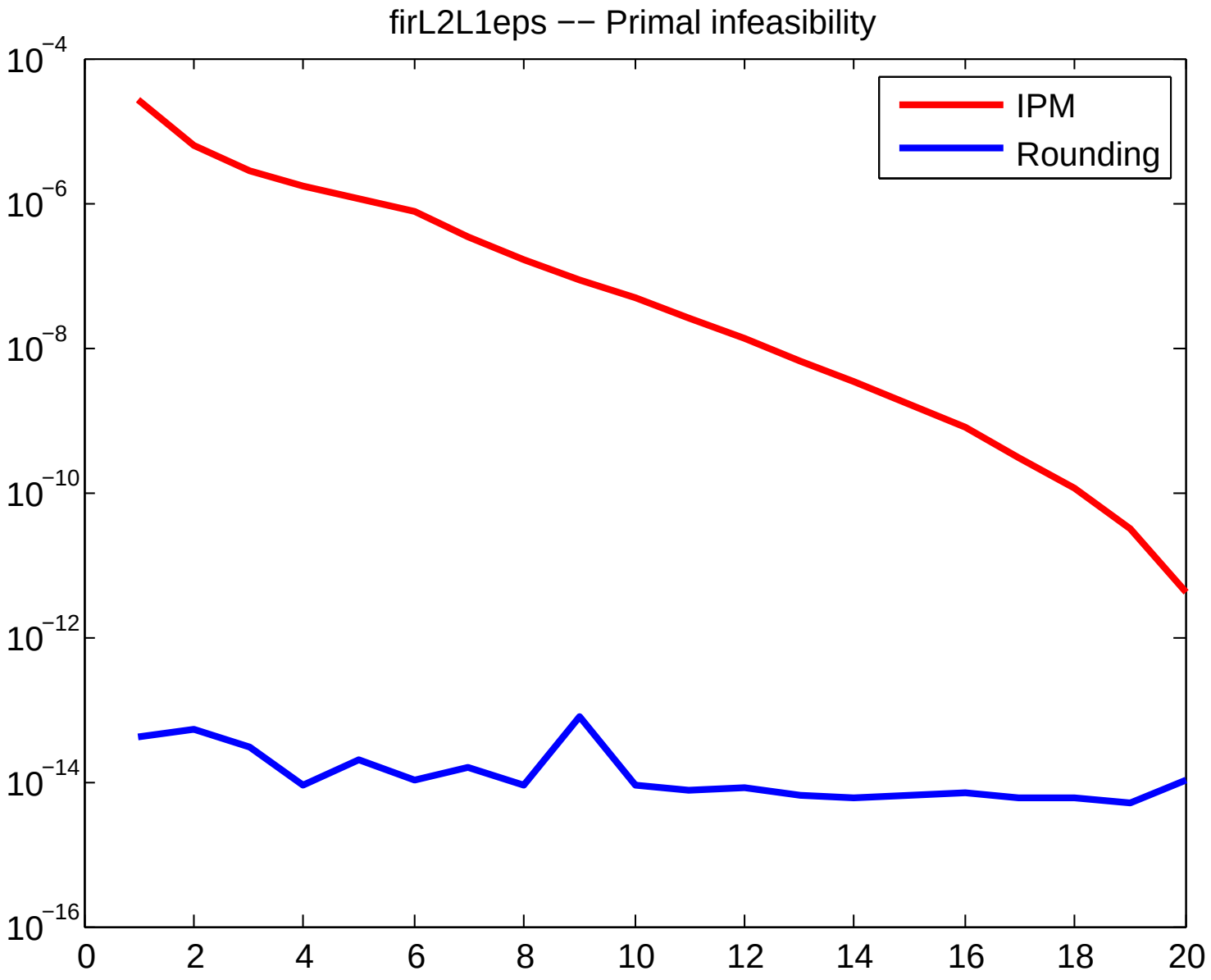
Computational results – the perfect



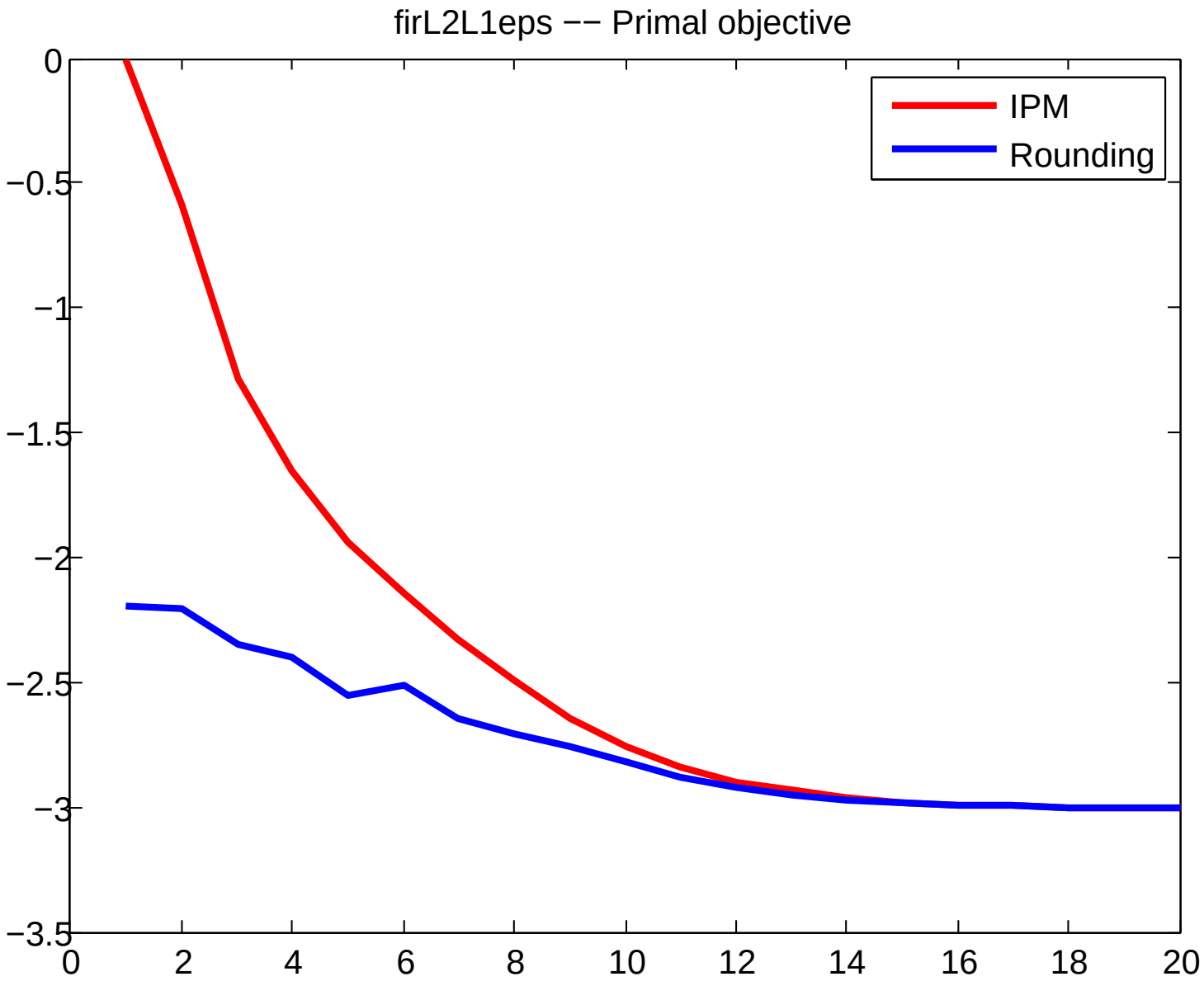
Computational results – the perfect



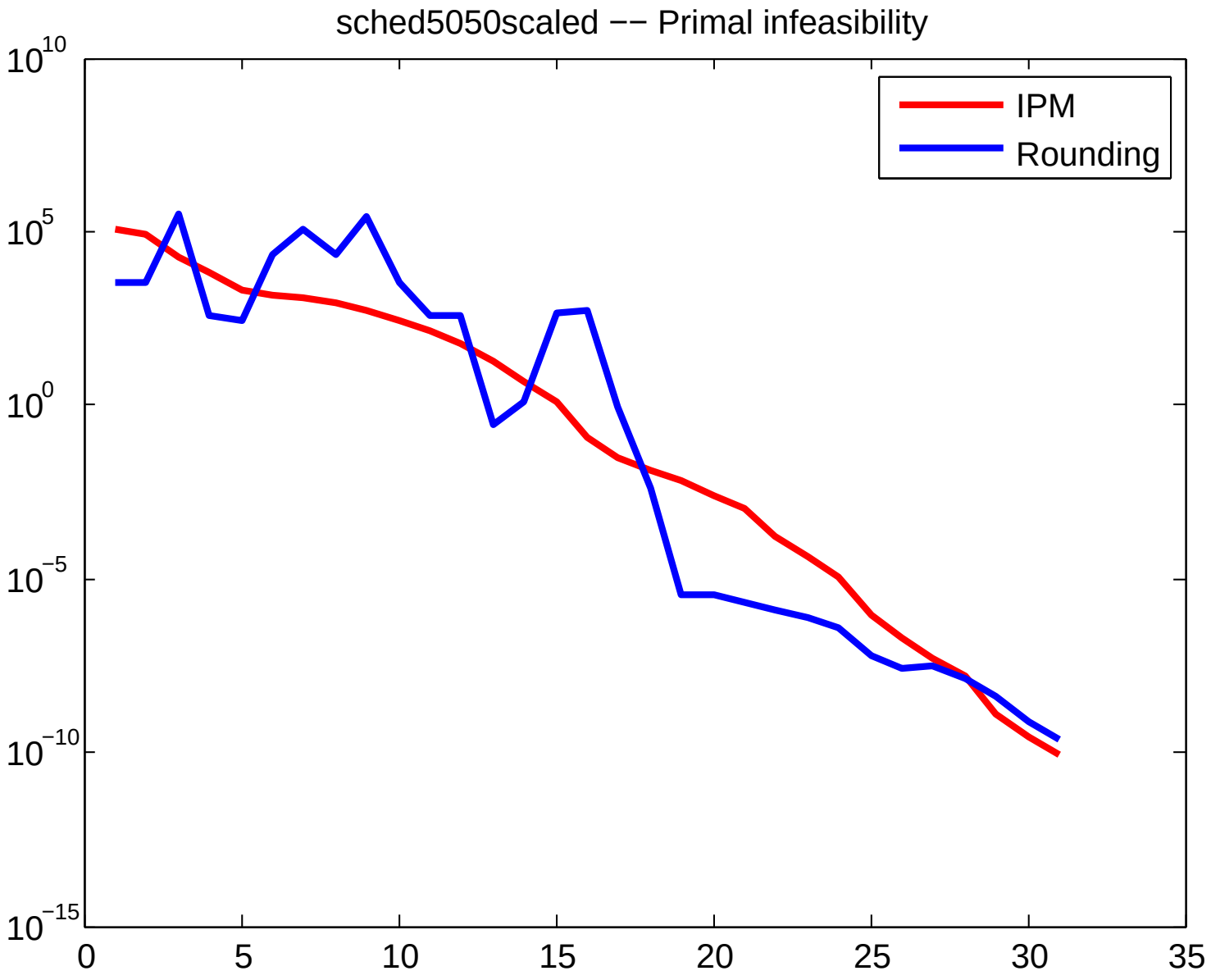
Computational results – the good



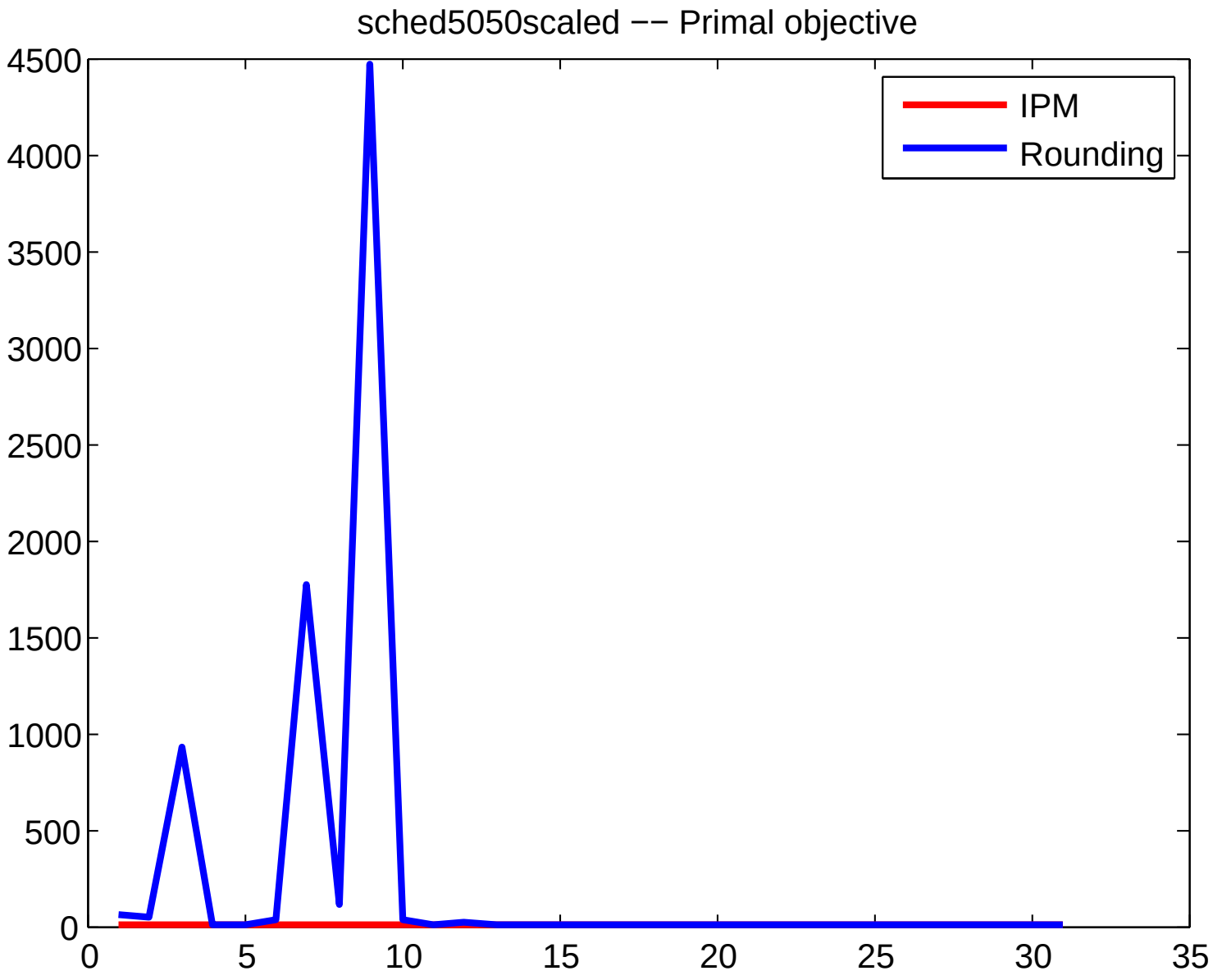
Computational results – the good



Computational results – the bad



Computational results – the bad



Interpretation and questions

- Rounded solutions are often feasible
- Rounded solutions can be optimal early
- Convergence of the Jordan frame
- Apply to SDP?
  - » eigenvectors are expensive
  - » small-dimensional SDP blocks (maybe from a decomposition?)
- Warmstarted rounding
- Emphasize complementarity in IPM
- Identify the cause of bad numerics in the rounding LPs
- Primal-dual rounding: combine Jordan frames
  - » 4 variables per cone

Questions?

imre@polik.net

Dual rounding

Decompose the dual solution

$$\mathbf{s^i} = (s_0^i, s^i) = \bar{\lambda}_+^i \left(\frac{1}{2}; u^i\right) + \bar{\lambda}_-^i \left(\frac{1}{2}; -u^i\right), \; i = 1, \ldots, K.$$

Dual rounding LP

$$\begin{aligned} \max \; & b^T y \\ y^T (A_0^i, A^i) \left(\frac{1}{2}; u^i\right) &\leq (c_0^i, c^i)^T \left(\frac{1}{2}; u^i\right), \; i = 1, \ldots, K, \\ y^T (A_0^i, A^i) \left(\frac{1}{2}; -u^i\right) &\leq (c_0^i, c^i)^T \left(\frac{1}{2}; -u^i\right), \; i = 1, \ldots, K. \end{aligned}$$

Dual of dual rounding

$$
\begin{aligned}
\min \ & \sum_{i=1}^{K} \lambda_+^i (c_0^i, c^i)^T \left(\frac{1}{2}; u^i\right) + \sum_{i=1}^{K} \lambda_-^i (c_0^i, c^i)^T \left(\frac{1}{2}; -u^i\right) \\
& \sum_{i=1}^{K} \lambda_+^i (A_0^i, A^i) \left(\frac{1}{2}; u^i\right) + \sum_{i=1}^{K} \lambda_-^i (A_0^i, A^i) \left(\frac{1}{2}; -u^i\right) = b \\
& \lambda_+^i, \lambda_-^i \geq 0, \ i = 1, \ldots, K.
\end{aligned}
$$