IE 496 Midterm — Tuesday, February 24

Dr. Imre Pólik — (35+5 points)

The standard form of the semidefinite optimization problem is:

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0, \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0, \end{aligned}$$

where $X, C, S \in \mathbb{R}^{n\times n}$, $b, y \in \mathbb{R}^m$, and $\mathcal{A} : \mathbb{R}^{n\times n} \to \mathbb{R}^m$.

1. (10 points) For a vector $u \in \mathbb{R}^{n+1}$ define the determinant of u as $\det(u) = u_1^2 - \|u_{2:n}\|^2$.
   (a) Prove that $\det(u \circ u) = \det(u)^2$.
   (b) Show that $\det(\operatorname{Arr}(u)) = u_1^{n-2}\det(u)$, where $\operatorname{Arr}(u)$ is the arrowhead matrix of u.
2. (5 points) Consider the standard form of the primal-dual semidefinite optimization problem. Assume that the primal problem is feasible. Show that if C is positive definite, then there is no duality gap and the primal problem is solvable.
3. (10 points) Prove that the standard form semidefinite optimization problem always has an optimal solution of rank at most m (assuming of course that the optimum is attained). **Hints**: The optimal primal-dual solutions are simultaneously diagonalizable. For a linear programming problem with n nonnegative variables and m linear equalities there is always an optimal solution x where only at most m components (the basic variables) are nonzero.
4. (15 points) Consider the SDP problem

$$\begin{aligned} \min\ & C \bullet X \\ A \bullet X &= \beta \\ X &\succeq 0, \end{aligned}$$

   where $A \succ 0$ and $\beta > 0$. Show how to obtain an optimal solution using only two eigenvalue decompositions. What is the rank of the optimal solution you found?