IE 496 Final Exam — Due date: Thursday, April 30, 12 noon

Dr. Imre Pólik — (100+10 points)

This final exam contains 6 questions. You can use any material, but you should give proper reference for everything except for the two textbooks and your lecture notes or slides. Collaboration is not allowed, but if you have any questions, feel free to contact me.

The problems are from Mike Todd, Tamás Terlaky and myself.

Good luck!

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1. (Basics, 15 points) Show that the following are equivalent for symmetric matrices X and Y:
   (a) X and Y commute;
   (b) XY is symmetric;
   (c) X and Y can be simultaneously diagonalized, i.e., there is an orthogonal matrix Q such that Q^TXQ and Q^TYQ are both diagonal (to simplify the proof assume that X has distinct eigenvalues).

2. (IPM, 15 points) Let S and X be $n \times n$ symmetric positive definite matrices. Prove the following:
   (a) $\sup_X \{-S \bullet X + \ln\det(X)\} = -\ln\det(S) - n$
   (b) $n\ln(X \bullet S) - \ln\det(X) - \ln\det(S) \geq n\ln n$

3. (Duality, 15 points) Prove that either the primal or the dual feasible set is unbounded for the standard linear optimization problem. **Hint:** A convex set is unbounded if and only if it contains a half-line. Also see Gordan's theorem if you get stuck. Check if the same is true for semidefinite optimization.

4. (SOCP, 20 points) Prove the following: if $x \in \mathbb{R}^3$ is such that $x_1 + x_2 + x_3 \geq 0$ and

$$\left(x_1 - \frac{x_1+x_2+x_3}{3}\right)^2 + \left(x_2 - \frac{x_1+x_2+x_3}{3}\right)^2 + \left(x_3 - \frac{x_1+x_2+x_3}{3}\right)^2 \leq \frac{1}{2}\frac{(x_1+x_2+x_3)^2}{3}$$

   then $x_1, x_2, x_3 \geq 0$.
   Show that in general there is a constant C_n depending on the dimension of x such that if $\sum_{j=1}^n x_j \geq 0$ and

$$\sum_{i=1}^n \left(x_i - \frac{\sum_{j=1}^n x_j}{n}\right)^2 \leq C_n \frac{\left(\sum_{j=1}^n x_j\right)^2}{n},$$

   then $x_1, \ldots, x_n \geq 0$. For some bonus points give an explicit formula for the best possible value of C_n. **Hint:** What are the sets defined by these quadratic inequalities?

5. (SDP, 30 points) Consider the following semidefinite programming problem:

$$\begin{aligned} \min C \bullet X \\ AX + XA = B \\ X \succeq 0, \end{aligned}$$

where $A, B, C, X \in \mathbb{R}^{n \times n}$.

   (a) Use the special form of operator $\mathcal{A}$ to streamline the computation of the Newton system, analyze the primal and dual HKM scalings.
   If n is the dimension of the matrices, then the Newton system is of size $n^2 \times n^2$ and is typically fully dense, and then it would take n^6 work to factorize it. For this reason iterative solution methods are popular for these Newton systems. These methods sole a linear system $Mu = p$ by subsequently applying M to vectors, so we never have to formulate M, we just need to be able to compute Mu for a given u. This could be a lot cheaper then forming M, but on the other hand, these methods can't provide very accurate solutions. In our case this means that instead of computing $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$, we compute $\mathcal{A}(\mathcal{E}^{-1}(\mathcal{F}(\mathcal{A}^*Y)))$.

   (b) Find the computational cost of applying $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ to a matrix Y for NT and primal/dual HKM scaling.

6. (Modelling, 15 points) Formulate the SDP relaxation of the following problem. Given a graph G with $n = 2k$ nodes find the cut that contains the fewest possible edges and divides the graph into two subsets whose cardinality differs by at most p.