Problems 1-3 are from Mike Todd.

1. (7 points) Suppose that instead of replacing the nonnegative vector \( x \) with a positive semidefinite matrix \( X \) in the standard-form LP, we replace each individual nonnegative variable by a positive semidefinite matrix. We might then get the problem (studied by Bellman and Fan in 1963!)

\[
\begin{align*}
\min & \quad \sum_{j=1}^{n} C_j \cdot X_j \\
\text{subject to} & \quad \sum_{j=1}^{n} A_{ij} \cdot X_j = b_i, \forall i = 1,\ldots,m \\
& \quad X_j \succeq 0, \forall j = 1,\ldots,n.
\end{align*}
\]

Show how this can be formulated as an SDP.

2. (7 points) Suppose that \( X \) is positive definite. Show that \( X - vv^T \succeq 0 \) if and only if \( v^TX^{-1}v \leq 1 \).

3. (6 points) Show that the second-order constraint \( \|Ay\|_2 \leq c^Tx + d \) can be written using semidefinite constraints.

4. (10 points) Consider the following linear operator:

\[
\mathcal{A} : \mathbb{R}^{2 \times 2} \to \mathbb{R}^2 \\
\mathcal{A}X = AXe + X Ae,
\]

where \( A \) and \( X \) are \( 2 \times 2 \) symmetric matrices and \( e = (1,1) \). Find the matrices \( A_1 \) and \( A_2 \) such that \( \mathcal{A}X = (A_1 \cdot X, A_2 \cdot X) \). For some extra credit, generalize the result to the case when \( X \) and \( A \) are \( n \times n \) matrices.