

Problems 1-3 are from Mike Todd.

- (7 points) Suppose that instead of replacing the nonnegative vector  $x$  with a positive semidefinite matrix  $X$  in the standard-form LP, we replace each individual nonnegative variable by a positive semidefinite matrix. We might then get the problem (studied by Bellman and Fan in 1963!)

$$\begin{aligned} \min \quad & \sum_{j=1}^n C_j \bullet X_j \\ & \sum_{j=1}^n A_{ij} \bullet X_j = b_i, \forall i = 1, \dots, m \\ & X_j \succeq 0, \forall j = 1, \dots, n. \end{aligned}$$

Show how this can be formulated as an SDP.

- (7 points) Suppose that  $X$  is positive definite. Show that  $X - vv^T \succeq 0$  if and only if  $v^T X^{-1} v \leq 1$ .
- (6 points) Show that the second-order constraint  $\|Ay\|_2 \leq c^T x + d$  can be written using semidefinite constraints.
- (10 points) Consider the following linear operator:

$$\begin{aligned} \mathcal{A} : \mathbb{R}^{2 \times 2} &\rightarrow \mathbb{R}^2 \\ \mathcal{A}X &= AXe + XAe, \end{aligned}$$

where  $A$  and  $X$  are  $2 \times 2$  symmetric matrices and  $e = (1, 1)$ . Find the matrices  $A_1$  and  $A_2$  such that  $\mathcal{A}X = (A_1 \bullet X, A_2 \bullet X)$ . For some extra credit, generalize the result to the case when  $X$  and  $A$  are  $n \times n$  matrices.