An introduction to cone optimization software

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Outline

1. Conic optimization
   - Problem definition and examples
   - Algorithms
2. State-of-the-art
   - What's available?
   - What's missing?
   - What's going on?

Problem definition

- Primal-dual form

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\in \mathcal{K} \end{aligned} \qquad \begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\in \mathcal{K}^* \end{aligned}$$

- Standard cones
  - LP $\mathcal{K} = \mathbb{R}^n_+$
  - SOCP $\mathcal{K} = \{(x_0, x) \in \mathbb{R}_+ \times \mathbb{R}^n : x_0 \geq \|x\|_2\}$
  - SDP $\mathcal{K} = \{X \in \mathbb{R}^{n\times n} : X \succeq 0\}$
  - products
- Exotic cones
  - intersections (PSD and nonnegative)
  - complex cones
  - homogeneous cones
  - SOS, nonnegative polynomials

SDP is different

$$\begin{aligned} \min \ & \mathrm{Tr}(CX) & \max \ & b^T y \\ \mathrm{Tr}(A_i X) &= b_i, i = 1, \dots, m & \sum_{i=1}^m A_i y_y + S &= C \\ X &\succeq 0 & S &\succeq 0 \end{aligned}$$

- Very strict format
- S inherits structure from A_i, C
- X doesn't
- Low rank A_i simplifies

$$A_i = a_i a_i^T \Rightarrow \mathrm{Tr}(A_i X) = a_i^T X a_i$$

Interior point methods

- Well-established (Nesterov-Nemirovski, Renegar)
- Widely implemented
- Excellent iteration complexity
- Costly iterations
- Overall complexity
  (dense SDP with m equalities, $n \times n$ matrices):

$$\mathcal{O}\left((mn^3 + m^2n^2 + m^3)\sqrt{n}\log(1/\varepsilon)\right)$$

- CSDP, DSDP, SDPA, SDPT3, SeDuMi, Mosek, CVXOPT, CPLEX (SOCP)

Nonlinear optimization methods

- General NLP (LOQO, PenSDP, fmincon)
- Spectral bundle methods (SBmeth)
- First order methods (?)
- How to input the cone? (OK for SOCP)
- Differentiability? (SOCP)
- Structure is usually lost
- Usually limited size

What's available? – I

- Good general SDP/SOCP solvers
  - state-of-the-art for dense problems
  - number of equalities, dimension ≤ 8000 (dense!)
  - final precision: 10^{-9}
- Some structured solvers
  - sparsity handling (limited by Matlab)
  - low rank coefficients (SPDLR, SDPT3)
- Implementations in C/C++/Matlab
- Problem libraries (SDPLIB, DIMACS, ...)
- Benchmarking (Hans Mittelmann)

What's available? – II

- Parallelization
  - OpenMP (CSDP)
  - MPI (SDPA)
  - Matlab
- Some preprocessing
  - (block) diagonal structure (SeDuMi)
  - matrix completion? (SDPA)
- Some modelling languages
  - CVX
  - YALMIP
  - ((GAMS))

What's missing? – I

- Better algorithms
  - Simplex-type method?
  - ?
- Preprocessing
  - mixed LP/SOCP/SDP problems
  - decomposition
  - generalization of LP techniques
- Special treatment of cone intersections

What's missing? – II

- Preprocessing
  - decomposition
  - detecting special structure
- More cones
- Embeddability
  - CSDP, SDPA for SDP
- Interfaces
- Integer conic optimization
- Support from major modelling languages
  - GAMS for SOCP
- All of the above in one!

What's going on?

- Preprocessing
  - solver or modelling language?
  - decomposition
  - special input formats (SDPLR)
  - rescaling
- Streamlined linear algebra
  - fixed/unconstrained variables
  - symbolic tools, general linear operators (NCalgebra)
- Parallelization
  - mostly OpenMP

And the winner is...

- HUGE-scale problems: CSDP, SDPA
- Ease of use: SeDuMi, SDPT3
- Speed: CSDP
- Accuracy: PENSDP, SeDuMi, SDPT3
- Largest background: SDPA
- Low rank coefficients: SDPLR
- Commercial solvers: CPLEX, LOQO, MOSEK, PENSDP
- Best overall: ?