Semidefinite programming
Interior-point methods: the Newton system, symmetrization

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Outline

1. The Newton system
   - A first try

2. Symmetrization
   - A simple method
   - The Monteiro-Zhang family

3. Computing the directions
   - Solving the system
   - Forming the system
The central path

- Optimality conditions
  \[ AX = b \]
  \[ A^*y + S = C \]
  \[ XS = 0 \]
  \[ X, S \succeq 0, \]

- Perturbation:
  \[ AX = b \]
  \[ A^*y + S = C \]
  \[ XS = \mu I \]
  \[ X, S \succ 0, \]

Exists if we have primal-dual strict feasibility, or equivalently, an interior point.
The Newton-system

\[
\begin{align*}
\mathcal{A} \Delta X &= 0 \\
\mathcal{A}^* \Delta y + \Delta S &= 0 \\
\Delta XS + X \Delta S &= \mu I - XS
\end{align*}
\]

The Newton-step

\[
\Delta S = -\mathcal{A}^* \Delta y = - \sum_{i=1}^{m} A_i \Delta y_i
\]

\[
\Delta X = \mu S^{-1} - X - X \Delta SS^{-1} \quad \text{not symmetric!}
\]

Force symmetry: no solution, overdetermined system

What went wrong? \((XS)\) is not necessarily symmetric
The AHO method

- Replace \( XS = \mu I \) with

\[
\frac{1}{2} (XS + SX) = \mu I,
\]

- The step is defined by:

\[
\begin{align*}
A \Delta X & = r_p \\
A^* \Delta y + \Delta S & = R_d \\
\frac{\Delta XS + S \Delta X}{2} + \frac{X \Delta S + \Delta SX}{2} & = R_M
\end{align*}
\]

- Not well-defined in general
The Monteiro–Zhang family

- Replace $XS = \mu I$ with

$$\frac{1}{2} \left( PXS P^{-1} + P^{-T} SXP^T \right) = \mu I,$$

- Alternatively, with $M = P^T P$:

$$\frac{1}{2} (MXS + SXM) = \mu M,$$

- $P$ may change in each iteration.

The step is defined by:

$$A \Delta X = r_p$$

$$A^* \Delta y + \Delta S = R_d$$

$$\frac{M \Delta XS + S \Delta XM}{2} + \frac{MX \Delta S + \Delta SXM}{2} = R_M$$
Some important members

- **AHO Alizadeh–Haeberly–Overton**
  - $P = M = I, \quad \frac{1}{2} (XS + SX) = \mu I$
  - Each MZ direction is a scaled AHO direction
  - $X' = PXP^T$

- **NT Nesterov–Todd**
  - $P = W^{-1/2}, \quad M = W^{-1}$
  - $WSW = X$, scaling matrix

- **HKM**
  - Helmberg–Rendl–Haeberly–Vanderbei–Wolkowicz
  - Kojima–Shindoh–Hara
  - Monteiro
  - $P = S^{1/2}, \quad M = S$

- **dual HKM**
  - $P = X^{-1/2}, \quad M = X^{-1}$
### Properties

- **Extends LP** Diagonal $X, S$: $S \Delta X + X \Delta S = \mu I - XS$
- **Predicts duality gap** $S \bullet \Delta X + X \bullet \Delta S = \mu n - X \bullet S$
- **Scale-invariance**
- **Well defined**
- **Primal-dual symmetry**

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<th>AHO</th>
<th>HKM</th>
<th>dual-HKM</th>
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Q: invariant under orthogonal transformation

See Mike Todd’s paper on all the symmetrization schemes.
Computing the directions I.

- General Newton-system

\[
\begin{align*}
A \Delta X &= r_p \\
A^* \Delta y + \Delta S &= R_d \\
E \Delta X + F \Delta S &= R_M
\end{align*}
\]

- Solution

\[
\begin{align*}
(A \mathcal{E}^{-1} F A^*) \Delta y &= \mathcal{E}^{-1} (R_{EF} - FR_d) \\
\Delta S &= R_d - A^* \Delta y \\
\Delta X &= \mathcal{E}^{-1} (R_{EF} - F \Delta S)
\end{align*}
\]

- How to find \( \mathcal{E}^{-1} \)?
- How much does it cost?
Computing the directions II.

- What is $\mathcal{E}^{-1}$ for MZ?

$$\mathcal{E} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$$

$$U \mapsto \frac{1}{2} (SUM + MUS)$$

- The resulting equation:

$$MUS + SUM = 2R$$

1. Compute $S^{\pm1/2} \Rightarrow M'U' + U'M' = 2R'$
2. Write $M'$ as $Q'D'Q'^T \Rightarrow \bar{D}\bar{U} + \bar{U}\bar{D} = 2\bar{R}$
3. $\bar{U}_{ij} = \frac{2\bar{R}_{ij}}{d_i + d_j}$

⇒ cheap! $O(n^3)$

- How to form $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$? How much does it cost?
Computing the directions III.

- Forming $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$

$$\mathbb{R}^m \xrightarrow{A^*} \mathbb{R}^{n^2} \xrightarrow{\mathcal{F}} \mathbb{R}^{n^2} \xrightarrow{\mathcal{E}^{-1}} \mathbb{R}^{n^2} \xrightarrow{\mathcal{A}} \mathbb{R}^m$$

$$(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)_{i,j} = A_i \cdot \mathcal{E}^{-1}\mathcal{F}A_j, \ i, j = 1, \ldots, m$$

- $\mathcal{E}^{-1}\mathcal{F}A_j$: (Cholesky, product) $\mathcal{O}(n^3)$, $m$-times
- $A_i \cdot \mathcal{E}^{-1}\mathcal{F}A_j$ (dot product) $\mathcal{O}(n^2)$, $m^2$-times

- Overall cost is $\mathcal{O}((m + n)mn^2) \Rightarrow$ expensive

- Total operation count:

$$\mathcal{O}\left((m + n)mn^{5/2} \log \left(\frac{1}{\varepsilon}\right)\right)$$

- Memory usage: $\mathcal{O}(mn^2 + m^2)$
- $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ is dense