

Semidefinite programming

Interior-point methods: the Newton system, symmetrization

Imre Pólik, PhD

Lehigh University
Department of Industrial and Systems Engineering

February 3, 2009

Outline

Outline

The Newton
system

Symmetrization

Computing the
directions

- 1 The Newton system
 - A first try
- 2 Symmetrization
 - A simple method
 - The Monteiro-Zhang family
- 3 Computing the directions
 - Solving the system
 - Forming the system

The central path

- Optimality conditions

$$\begin{array}{rcl}
 & \mathcal{A}X & = b \\
 \mathcal{A}^*y & & +S = C \\
 & XS & = 0 \\
 & X, S & \succeq 0,
 \end{array}$$

- Perturbation:

$$\begin{array}{rcl}
 & \mathcal{A}X & = b \\
 \mathcal{A}^*y & & +S = C \\
 & XS & = \mu I \\
 & X, S & \succ 0,
 \end{array}$$

Exists if we have primal-dual strict feasibility, or equivalently, an interior point.

Outline

The Newton system

A first try

Symmetrization

Computing the directions

The Newton-step

- The Newton-system

$$\begin{aligned} \mathcal{A}\Delta X &= 0 \\ \mathcal{A}^*\Delta y + \Delta S &= 0 \\ \Delta XS + X\Delta S &= \mu I - XS \end{aligned}$$

- The Newton-step

$$\Delta S = -\mathcal{A}^*\Delta y = -\sum_{i=1}^m A_i \Delta y_i$$

$$\Delta X = \mu S^{-1} - X - X\Delta S S^{-1} \quad \text{not symmetric!}$$

- Force symmetry: no solution, overdetermined system
- What went wrong? (XS is not necessarily symmetric)

The AHO method

- Replace $XS = \mu I$ with

$$\frac{1}{2} (XS + SX) = \mu I,$$

- The step is defined by:

$$\begin{aligned} \mathcal{A}\Delta X &= r_p \\ \mathcal{A}^*\Delta y + \Delta S &= R_d \\ \frac{\Delta XS + S\Delta X}{2} + \frac{X\Delta S + \Delta SX}{2} &= R_M \end{aligned}$$

- Not well-defined in general

Outline

The Newton
system

Symmetrization

A simple method
The Monteiro-Zhang
familyComputing the
directions

The Monteiro–Zhang family

- Replace $XS = \mu I$ with

$$\frac{1}{2} (PXS P^{-1} + P^{-T} S X P^T) = \mu I,$$

- Alternatively, with $M = P^T P$:

$$\frac{1}{2} (MXS + SXM) = \mu M,$$

- P may change in each iteration.
- The step is defined by:

$$\begin{aligned} \mathcal{A} \Delta X &= r_p \\ \mathcal{A}^* \Delta y + \Delta S &= R_d \\ \frac{M \Delta X S + S \Delta X M}{2} + \frac{M X \Delta S + \Delta S X M}{2} &= R_M \end{aligned}$$

Some important members

- AHO Alizadeh–Haeberly–Overton
 - $P = M = I, \quad \frac{1}{2} (XS + SX) = \mu I$
 - Each MZ direction is a scaled AHO direction
 - $X' = PXP^T$
- NT Nesterov–Todd
 - $P = W^{-1/2}, M = W^{-1}$
 - $WSW = X$, scaling matrix
- HKM
 - Helmberg–Rendl–Haeberly–Vanderbei–Wolkowicz
 - Kojima–Shindoh–Hara
 - Monteiro
 - $P = S^{1/2}, M = S$
- dual HKM
 - $P = X^{-1/2}, M = X^{-1}$

Properties

- Extends LP Diagonal $X, S: S\Delta X + X\Delta S = \mu I - XS$
- Predicts duality gap $S \bullet \Delta X + X \bullet \Delta S = \mu n - X \bullet S$
- Scale-invariance Take $X' = PXP^T$
- Well defined
- Primal-dual symmetry

Property	AHO	HKM	dual-HKM	NT
Extends LP	✓	✓	✓	✓
Predicts gap	✓	✓	✓	✓
Well-defined		✓	✓	✓
Scale invariant	Q	✓	✓	✓
PD-symmetric	✓			✓

Q: invariant under orthogonal transformation

See Mike Todd's paper on all the symmetrization schemes.

Outline

The Newton
system

Symmetrization

Computing the
directionsSolving the system
Forming the system

Computing the directions I.

- General Newton-system

$$\begin{array}{rcl} & \mathcal{A}\Delta X & = r_p \\ \mathcal{A}^* \Delta y & & + \Delta S = R_d \\ & \mathcal{E}\Delta X + \mathcal{F}\Delta S & = R_M \end{array}$$

- Solution

$$\begin{aligned} (\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*) \Delta y &= \mathcal{A}\mathcal{E}^{-1} (R_{EF} - \mathcal{F}R_d) \\ \Delta S &= R_d - \mathcal{A}^* \Delta y \\ \Delta X &= \mathcal{E}^{-1} (R_{EF} - \mathcal{F}\Delta S) \end{aligned}$$

- How to find \mathcal{E}^{-1} ?
- How much does it cost?

Computing the directions II.

- What is \mathcal{E}^{-1} for MZ?

$$\begin{aligned}\mathcal{E} : \mathbb{R}^{n \times n} &\rightarrow \mathbb{R}^{n \times n} \\ U &\mapsto \frac{1}{2}(SUM + MUS)\end{aligned}$$

- The resulting equation:

$$MUS + SUM = 2R$$

- 1 Compute $S^{\pm 1/2} \Rightarrow M'U' + U'M' = 2R'$
- 2 Write M' as $Q'D'Q'^T \Rightarrow \bar{D}\bar{U} + \bar{U}\bar{D} = 2\bar{R}$
- 3 $\bar{U}_{ij} = \frac{2\bar{R}_{ij}}{\bar{d}_i + \bar{d}_j}$

- \Rightarrow cheap! $\mathcal{O}(n^3)$
- How to form $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$? How much does it cost?

Computing the directions III.

- Forming $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$

$$\mathbb{R}^m \xrightarrow{\mathcal{A}^*} \mathbb{R}^{n^2} \xrightarrow{\mathcal{F}} \mathbb{R}^{n^2} \xrightarrow{\mathcal{E}^{-1}} \mathbb{R}^{n^2} \xrightarrow{\mathcal{A}} \mathbb{R}^m$$

$$(\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*)_{i,j} = A_i \bullet \mathcal{E}^{-1}\mathcal{F}A_j, \quad i, j = 1, \dots, m$$

- $\mathcal{E}^{-1}\mathcal{F}A_j$: (Cholesky, product) $\mathcal{O}(n^3)$, m -times
- $A_i \bullet \mathcal{E}^{-1}\mathcal{F}A_j$ (dot product) $\mathcal{O}(n^2)$, m^2 -times
- Overall cost is $\mathcal{O}((m+n)mn^2) \Rightarrow$ expensive
- Total operation count:

$$\mathcal{O}\left((m+n)mn^{5/2} \log\left(\frac{1}{\varepsilon}\right)\right)$$

- Memory usage: $\mathcal{O}(mn^2 + m^2)$
- $\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^*$ is dense