Semidefinite programming
Interior-point methods: the central path

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Outline

1. The central path
   - Orthogonality property
   - Perturbed complementarity conditions
   - Barrier functions
   - Primal and dual barrier problems
   - Properties of the central path

Orthogonality of feasible steps

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0 \end{aligned}$$

Lemma

If X_1, X_2 are primal feasible and S_1, S_2 are dual feasible, then $(X_1 - X_2) \bullet (S_1 - S_2) = 0$. In other words, the primal and dual feasible directions ΔX and ΔS are orthogonal.

The central path

- Optimality conditions

$$\begin{array}{rrrcl} & \mathcal{A}X & & = & b \\ \mathcal{A}^*y & & +S & = & C \\ & XS & & = & 0 \\ & X,\ S & & \succeq & 0, \end{array}$$

- Perturbation:

$$\begin{array}{rrrcl} & \mathcal{A}X & & = & b \\ \mathcal{A}^*y & & +S & = & C \\ & XS & & = & \mu I \\ & X,\ S & & \succ & 0, \end{array}$$

- Existence?
- Uniqueness?
- Analiticity?
- Convergence?

Optimality conditions for the barrier problem

Consider

$$\begin{aligned} \min\ & f(x) \\ Ax &= b \\ x &\geq 0, \end{aligned} \tag{NLP}$$

where $f(x)$ is a convex function.
Rewrite it as

$$\begin{aligned} \min\ & f(x) - \mu \sum_i \log x_i \\ Ax &= b, \end{aligned} \tag{NLP_μ}$$

where $\mu > 0$. Properties:

- The objective function is strictly convex if $\mu > 0$.
- As $\mu \to 0$ the optimal solutions of (NLP_μ) converge to an optimal solution of (NLP).

Barrier functions for semidefinite optimization

Consider

$$\phi(X) = -\ln\det(X)$$

Properties:

- $\phi(X) = -\sum_{i=1}^{n} \ln \lambda_i(X)$
- $\phi(X) < \infty \Leftrightarrow X \succ 0$
- $\phi(X) \to \infty$ if X is approaching the boundary of $\mathbb{S}_{+}^{n\times n}$
- $\phi(X)$ is differentiable

$$\begin{aligned} \phi'(X) &= -X^{-1} \\ \phi''(X)[U,V] &= (X^{-1}UX^{-1}) \bullet V \\ \phi'''(X)[U,V,W] &= \dots \end{aligned}$$

- $\phi(X)$ is strictly convex

SDP with a barrier

Primal barrier problem:

$$
\begin{aligned}
\min C \bullet X - \mu \ln \det(X) \\
\mathcal{A}X = b
\end{aligned}
$$

Dual barrier problem:

$$
\max \; b^T y + \mu \ln \det(C - \mathcal{A}^* y)
$$

Optimality conditions:

$$
\begin{array}{rcccl}
 & \mathcal{A}X & & = & b \\
\mathcal{A}^* y & & +S & = & C \\
 & XS & & = & \mu I \\
 & X,\, S & & \succ & 0,
\end{array}
$$

Properties of the central path

If both the primal and dual problems are strictly feasible, then

- The central path exists.
- The central path is unique.
- The central path is an analytic curve.
- As $\mu \to 0$, the central path converges to a maximally complementary optimal solution.
- If strict complementarity holds, then the central path converges to the analytic center of the optimal set.

The idea behind interior-point methods is to follow the central path.