

# Semidefinite programming

## Optimality, complementarity, degeneracy

Imre Pólik, PhD

Lehigh University  
Department of Industrial and Systems Engineering

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# Outline

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Obtaining the dual

Optimality

Self-dual  
embedding

### 1 Obtaining the dual

### 2 Optimality

- Complementarity
- Strict complementarity
- Maximal complementarity

### 3 Self-dual embedding

- Ensuring strict feasibility

# The dual is really the dual

- Get a lower bound for

$$\min C \bullet X$$

$$\mathcal{A}X = b$$

$$X \succeq 0$$

- Multiplier  $y \in \mathbb{R}^m$ , s.t.  $y^T(\mathcal{A}X) \leq C \bullet X, \forall X \succeq 0$ .
- $y^T(\mathcal{A}X) = (\mathcal{A}^*y) \bullet X$
- $(C - \mathcal{A}^*y) \bullet X \geq 0, \forall X \succeq 0$
- $C - \mathcal{A}^*y \succeq 0$
- Best lower bound:

$$\max b^T y$$

$$\mathcal{A}^*y + S = C$$

$$S \succeq 0$$

# Optimality

- Zero duality gap implies optimality:

$$\begin{array}{rcl}
 \mathcal{A}X & = & b \\
 \mathcal{A}^*y + S & = & C \\
 XS & = & 0 \quad (\Leftrightarrow \text{Tr}(XS) = 0) \\
 X, S & \succeq & 0,
 \end{array}$$

- Only sufficient condition
- Necessary under strict feasibility
- Complementarity:  $XS = 0$
- Strict complementarity:  $X + S \succ 0$

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# Strict complementarity?

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$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Optimal solutions:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$XS = 0$ , but  $X + S \neq 0$ : *strict* complementarity does not hold.

# Maximal complementarity

## Definition

The solutions  $X$  and  $S$  are maximally complementary if they have the largest possible rank.

## Theorem (Optimal solutions)

Any primal optimal solution is written as  $X = Q_B U_X Q_B^T$ , where

- The  $r$  columns of  $Q_B$  form a basis of  $\mathcal{R}(X)$
- $Q_B^T Q_B = I$  (orthonormal basis)
- $U_X \succeq 0$ ,  $r \times r$  matrix
- $U_X \succ 0$  if  $X$  is in the relative interior of the optimal set.

Similarly for dual.

# Self-dual embedding model for SDP

- Strict feasibility needed

$$\begin{array}{rcccccl}
 & & & & \min & (x_0^T s_0 + 1)\theta \\
 & & Ax & -b\tau & +\bar{b}\theta & = 0 \\
 -A^T y & & & +c\tau & -\bar{c}\theta & = s \\
 b^T y & -c^T x & & & +\bar{z}\theta & = \kappa \\
 -\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T \tau & & & = -x_0^T s_0 - 1 \\
 & & & & & x \succeq 0, \tau \geq 0, s \succeq 0, \kappa \geq 0,
 \end{array}$$

where

- $x_0, s_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, \tau, \theta \in \mathbb{R}$
- $\bar{b} = b - Ax_0, \bar{c} = c - A^T y_0 - s_0, \bar{z} = c^T x_0 - b^T y_0 + 1$
- $\tau > 0, \kappa = 0$ :  $x/\tau, y/\tau, s/\tau$  are optimal
- $\tau = 0, \kappa > 0$ : infeasibility
- $\tau = 0, \kappa = 0$ : ? (no strictly complementary solutions exist, no improving rays exist)