Semidefinite programming
Optimality, complementarity, degeneracy

Imre Pólik, PhD

Lehigh University
Department of Industrial and Systems Engineering

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Outline

The dual is really the dual

- Get a lower bound for

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0 \end{aligned}$$

- Multiplier $y \in \mathbb{R}^m$, s.t. $y^T(\mathcal{A}X) \leq C \bullet X$, $\forall X \succeq 0$.
- $y^T(\mathcal{A}X) = (\mathcal{A}^* y) \bullet X$
- $(C - \mathcal{A}^* y) \bullet X \geq 0$, $\forall X \succeq 0$
- $C - \mathcal{A}^* y \succeq 0$
- Best lower bound:

$$\begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0 \end{aligned}$$

Optimality

- Zero duality gap implies optimality:

$$\begin{array}{rrcll} & \mathcal{A}X & = & b & \\ \mathcal{A}^* y & +S & = & C & \\ & XS & = & 0 & (\Leftrightarrow \mathrm{Tr}(XS) = 0) \\ & X,\ S & \succeq & 0, & \end{array}$$

- Only sufficient condition
- Necessary under strict feasibility
- Complementarity: $XS = 0$
- Strict complementarity: $X + S \succ 0$

Strict complementarity?

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Optimal solutions:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$XS = 0$, but $X + S \not\succ 0$: *strict* complementarity does not hold.

Maximal complementarity

Definition

The solutions X and S are maximally complementary if they have the largest possible rank.

Theorem (Optimal solutions)

Any primal optimal solution is written as $X = Q_B U_X Q_B^T$, *where*

- *The* r *columns of* Q_B *form a basis of* $\mathcal{R}(X)$
- $Q_B^T Q_B = I$ *(orthonormal basis)*
- $U_X \succeq 0$, $r \times r$ *matrix*
- $U_X \succ 0$ *if* X *is in the relative interior of the optimal set.*

Similarly for dual.

Self-dual embedding model for SDP

- Strict feasibility needed

$$
\begin{array}{rrrrcl}
\multicolumn{6}{c}{\min\ (x_0^T s_0 + 1)\theta} \\
& Ax & -b\tau & +\bar{b}\theta & = & 0 \\
-A^T y & & +c\tau & -\bar{c}\theta & = & s \\
b^T y & -c^T x & & +\bar{z}\theta & = & \kappa \\
-\bar{b}^T y & -\bar{c}^T x & -\bar{z}^T \tau & & = & -x_0^T s_0 - 1 \\
\multicolumn{6}{c}{x \succeq 0,\ \tau \geq 0,\ s \succeq 0,\ \kappa \geq 0,}
\end{array}
$$

  where

  - $x_0, s_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$, $\tau, \theta \in \mathbb{R}$
  - $\bar{b} = b - Ax_0$, $\bar{c} = c - A^T y_0 - s_0$, $\bar{z} = c^T x_0 - b^T y_0 + 1$
- $\tau > 0$, $\kappa = 0$: $x/\tau, y/\tau, s/\tau$ are optimal
- $\tau = 0$, $\kappa > 0$: infeasibility
- $\tau = 0$, $\kappa = 0$: ? (no strictly complementary solutions exist, no improving rays exist)