

# Semidefinite programming

## Feasibility and duality

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# Outline

Outline

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Duality

## 1 Feasibility

- Review of LP
- Weak and strong infeasibility
- Farkas lemma for SDP

## 2 Duality

- Review of LP duality
- SDP duality

# Farkas lemma

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$$\min c^T x$$

$$Ax = b$$

$$x \geq 0$$

$$\max b^T y$$

$$A^T y + s = c$$

$$s \geq 0$$

## Theorem (Farkas lemma)

*The following two statements are equivalent:*

- 1 *There is an  $x \in \mathbb{R}^n$  such that  $Ax = b$  and  $x \geq 0$ .*
- 2 *There is no  $y \in \mathbb{R}^m$  such that  $A^T y \leq 0$  and  $b^T y = 1$ .*

*In other words, primal (dual) infeasibility is equivalent to the existence of a dual (primal) improving direction.*

# Weak infeasibility for SDP

Consider

$$\min \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \max y_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 1 \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y_1 + S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X \succeq 0 \quad S \succeq 0$$

- The dual is infeasible.
- There is no primal improving direction.

But:

- The dual is *almost* feasible.
- There is a primal *almost* improving direction.

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## Weak infeasibility

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$$\min C \bullet X$$

$$\mathcal{A}X = b$$

$$X \succeq 0$$

$$\max b^T y$$

$$\mathcal{A}^*y + S = C$$

$$S \succeq 0$$

The following are equivalent if the primal is infeasible:

- 1 For every  $\varepsilon > 0$  there is an  $X \succeq 0$  such that  $\|\mathcal{A}X - b\| \leq \varepsilon$ .
- 2 For every  $\varepsilon > 0$  there is a  $y \in \mathbb{R}^m$  and  $S \succeq 0$  such that  $\|\mathcal{A}^*y + S\| \leq \varepsilon$  and  $b^T y = 1$ .

The following are equivalent if the dual is infeasible:

- 1 For every  $\varepsilon > 0$  there is a  $y \in \mathbb{R}^m$  and  $S \succeq 0$  such that  $\|\mathcal{A}^*y + S - C\| \leq \varepsilon$ .
- 2 For every  $\varepsilon > 0$  there is an  $X \succeq 0$  such that  $\|\mathcal{A}X\| \leq \varepsilon$  and  $C \bullet X = -1$ .

## Farkas lemma for SDP

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$$\min C \bullet X$$

$$\mathcal{A}X = b$$

$$X \succeq 0$$

$$\max b^T y$$

$$\mathcal{A}^* y + S = C$$

$$S \succeq 0$$

## Theorem (Farkas lemma for SDP)

The following two statements are NOT equivalent, but

1  $\Rightarrow$  2:

- 1 There is an  $X \in \mathbb{R}^{n \times n}$  such that  $\mathcal{A}X = b$  and  $X \succeq 0$ .
- 2 There is no  $y \in \mathbb{R}^m$  such that  $\mathcal{A}^* y \preceq 0$  and  $b^T y = 1$ .

In other words, the existence of a primal (dual) improving direction implies dual (primal) infeasibility. The converse fails due to weak infeasibility.

# LP duality

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$$\min c^T x$$

$$Ax = b$$

$$x \geq 0$$

$$\max b^T y$$

$$A^T y + s = c$$

$$s \geq 0$$

## Theorem (Duality of LP)

- *If the primal (dual) problem is unbounded, then the dual (primal) is infeasible.*
- *If the primal (dual) problem is infeasible, then the dual (primal) problem is unbounded or infeasible.*
- *If both problems are feasible, then the optimal values are the same (there is no duality gap), and the optimum is attained for both problems.*

# Example 1, duality gap

$$\min \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \bullet X = 1$$

$$X \succeq 0$$

- Dual optimum is 0, with  $y = 0$

- Primal optimum is  $\alpha$ , with  $x_{11} = 1$

$$\max y_2$$

$$y_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + S = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S \succeq 0$$

What went wrong: dual feasible set is too small (Slater condition).

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## Example 2, unattained optimum

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$$\min \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \max 2y_1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 2 \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} y_1 + S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$X \succeq 0 \quad S \succeq 0$$

- Dual optimal value is 0, optimal dual solution is  $y_1 = 0$ .
- Primal optimal value is 0, but no optimal primal solution.

What went wrong: dual feasible set is too small (Slater condition).

# SDP duality

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$$\min C \bullet X$$

$$\mathcal{A}X = b$$

$$X \succeq 0$$

$$\max b^T y$$

$$\mathcal{A}^* y + S = C$$

$$S \succeq 0$$

## Theorem (Duality for SDP)

- *If the primal (dual) problem is strictly feasible, then the dual (primal) problem is solvable and the optimal values are equal.*
- *If both problems are strictly feasible, then they are both solvable and the optimal values are equal.*