Semidefinite programming
Feasibility and duality

Imre Pólik, PhD

Lehigh University
Department of Industrial and Systems Engineering

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Outline

1. Feasibility
   - Review of LP
   - Weak and strong infeasibility
   - Farkas lemma for SDP
2. Duality
   - Review of LP duality
   - SDP duality

Farkas lemma

$$\min c^T x \qquad \max b^T y$$

$$Ax = b \qquad A^T y + s = c$$

$$x \geq 0 \qquad s \geq 0$$

Theorem (Farkas lemma)

The following two statements are equivalent:

1. *There is an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.*
2. *There is no $y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y = 1$.*

In other words, primal (dual) infeasibility is equivalent to the existence of a dual (primal) improving direction.

Weak infeasibility for SDP

Consider

$$\min \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad\qquad \max y_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 1 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y_1 + S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X \succeq 0 \qquad\qquad S \succeq 0$$

- The dual is infeasible.
- There is no primal improving direction.

But:

- The dual is *almost* feasible.
- There is a primal *almost* improving direction.

Weak infeasibility

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0 \end{aligned} \qquad \begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0 \end{aligned}$$

The following are equivalent if the primal is infeasible:

1. For every $\varepsilon > 0$ there is an $X \succeq 0$ such that $\|\mathcal{A}X - b\| \leq \varepsilon$.
2. For every $\varepsilon > 0$ there is a $y \in \mathbb{R}^m$ and $S \succeq 0$ such that $\|\mathcal{A}^* y + S\| \leq \varepsilon$ and $b^T y = 1$.

The following are equivalent if the dual is infeasible:

1. For every $\varepsilon > 0$ there is a $y \in \mathbb{R}^m$ and $S \succeq 0$ such that $\|\mathcal{A}^* y + S - C\| \leq \varepsilon$.
2. For every $\varepsilon > 0$ there is an $X \succeq 0$ such that $\|\mathcal{A}X\| \leq \varepsilon$ and $C \bullet X = -1$.

Farkas lemma for SDP

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0 \end{aligned}$$

Theorem (Farkas lemma for SDP)

The following two statements are NOT equivalent, but $1 \Rightarrow 2$*:*

1. *There is an* $X \in \mathbb{R}^{n\times n}$ *such that* $\mathcal{A}X = b$ *and* $X \succeq 0$.
2. *There is no* $y \in \mathbb{R}^m$ *such that* $\mathcal{A}^* y \preceq 0$ *and* $b^T y = 1$.

In other words, the existence of a primal (dual) improving direction implies dual (primal) infeasibility. The converse fails due to weak infeasibility.

LP duality

$$\begin{aligned} \min\ & c^T x \\ Ax &= b \\ x &\geq 0 \end{aligned} \qquad \begin{aligned} \max\ & b^T y \\ A^T y + s &= c \\ s &\geq 0 \end{aligned}$$

Theorem (Duality of LP)

- *If the primal (dual) problem is unbounded, then the dual (primal) is infeasible.*
- *If the primal (dual) problem is infeasible, then the dual (primal) problem is unbounded or infeasible.*
- *If both problems are feasible, then the optimal values are the same (there is no duality gap), and the optimum is attained for both problems.*

Example 1, duality gap

$$\min \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet X = 0$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \bullet X = 1$$
$$X \succeq 0$$

- Dual optimum is 0, with $y = 0$
- Primal optimum is α, with $x_{11} = 1$

$$\max y_2$$
$$y_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + S = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$S \succeq 0$$

What went wrong: dual feasible set is too small (Slater condition).

Example 2, unattained optimum

$$\min \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad \max\ 2y_1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \bullet \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 2 \qquad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} y_1 + S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$X \succeq 0 \qquad S \succeq 0$$

- Dual optimal value is 0, optimal dual solution is $y_1 = 0$.
- Primal optimal value is 0, but no optimal primal solution.

What went wrong: dual feasible set is too small (Slater condition).

SDP duality

$$\begin{aligned} \min\ & C \bullet X \\ \mathcal{A}X &= b \\ X &\succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ & b^T y \\ \mathcal{A}^* y + S &= C \\ S &\succeq 0 \end{aligned}$$

Theorem (Duality for SDP)

- *If the primal (dual) problem is strictly feasible, then the dual (primal) problem is solvable and the optimal values are equal.*
- *If both problems are strictly feasible, then they are both solvable and the optimal values are equal.*