Semidefinite programming basic concepts

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Notation

$\mathbb{S}^n$: $n \times n$ symmetric matrices

$\mathbb{S}^{n\times n}_+$: $n \times n$ symmetric, positive semidefinite matrices

$\mathbb{S}^{n\times n}_{++}$: $n \times n$ symmetric, positive definite matrices

$U \succeq 0$: U is symmetric positive semidefinite

$U \succ 0$: U is symmetric positive definite

$U \succeq V$: $U - V$ is symmetric positive semidefinite

$U \succ V$: $U - V$ is symmetric positive definite

Semidefinite matrices

The following are equivalent ($M \in \mathbb{R}^{n\times n}$):

- $M \succeq 0$ (M is symmetric, positive semidefinite)
- $x^T M x \geq 0,\ \forall x \in \mathbb{R}^n$
- $\lambda_i(M) \geq 0,\ \forall i = 1, \dots, n$ (all the eigenvalues of M are nonnegative)
- $\det(M_{I,I}) \geq 0,\ \forall I \subseteq \{1, \dots, n\}$ (all the principal minors are nonnegative)
- $M = LL^T$ for some lower triangular matrix L (Cholesky factorization)
- $M = QDQ^T$, D is nonnegative diagonal, $QQ^T = I$
- $M = \sum_i \lambda_i v_i v_i^T$, $\lambda_i \geq 0$
- $M = M^{1/2} M^{1/2}$ (square root), $M^{1/2} = \sum_i \sqrt{\lambda_i} v_i v_i^T$

The scalar product

$U, V \in \mathbb{R}^{n\times n}$, $U \bullet V = \mathrm{Tr}(U^T V)$

Properties:

- It really defines a scalar product
  - $U \bullet U \geq 0$, and $U \bullet U = 0$ implies $U = 0$
  - $U \bullet V = V \bullet U$ (symmetry)
  - $(\alpha U \bullet V) = \alpha(U \bullet V)$, $(U + W) \bullet V = U \bullet V + W \bullet V$ (linearity)
- $\mathrm{Tr}(UV) = \mathrm{Tr}(VU) = \mathrm{Tr}(U^T V^T) = \mathrm{Tr}(V^T U^T) = \sum_i \sum_j U_{ij} V_{ji}$
- If $QQ^T = I$ (orthogonal), then $(QUQ^T) \bullet (QVQ^T) = U \bullet V$
- If $U, V \succeq 0$, then $U \bullet V \geq 0$, and $U \bullet V = 0$ if and only if $UV = 0$

Other properties

- $uu^T \succeq 0$
- If $U \succeq 0$, then $U_{ii} \geq 0$, and $U_{ii} = 0$ implies $U_{ik} = U_{ki} = 0$, $\forall k = 1, \ldots, n$
- If $U \succeq 0$, then $PUP^T \succeq 0$
- If $U \succeq 0$, then every principal submatrix of U is PSD.
- $x^T U x = \mathrm{Tr}(x^T Q x) = \mathrm{Tr}(Q x x^T) = Q \bullet x x^T$
- If A is positive definite, then $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0$ if and only if $C - B^T A^{-1} B \succeq 0$ (Schur complement)
- U, V symmetric. The following are equivalent:
  - $UV = VU$
  - UV is symmetric
  - U and V are simultaneously diagonalizable, $U = PD_U P^T, V = PD_V P^T$

Linear operators over matrices

$X \in \mathbb{R}^{n\times n}$, $\mathcal{A} : \mathbb{R}^{n\times n} \to \mathbb{R}^m$

- $\mathcal{A}X = (A_i \bullet X)_{i=1}^m$
- $\mathcal{A}^* y = \sum_i y_i A_i$
- This is really an adjoint!

$$\begin{aligned}(\mathcal{A}X)^T y = \left((A_i \bullet X)_{i=1}^m\right)^T y &= \sum_i y_i (X \bullet A_i) \\ &= \sum_i (X \bullet y_i A_i) = \sum_i X \bullet (y_i A_i) \\ &= X \bullet (\mathcal{A}^* y)\end{aligned}$$

Convex cones

Given $x, y \in \mathcal{C} \subseteq \mathbb{R}^n$, $\alpha \geq 0$, $0 \leq \lambda \leq 1$

cone: $\alpha x \in \mathcal{C}$

convex: $\lambda x + (1-\lambda) y \in \mathcal{C}$

Dual cone: $\mathcal{C}^* = \left\{ v \in \mathbb{R}^n : x^T v \geq 0, \forall x \in \mathcal{C} \right\}$

$\mathbb{S}_+^{n\times n}$ is

- convex cone
- self-dual, $\left(\mathbb{S}_+^{n\times n}\right)^* = \mathbb{S}_+^{n\times n}$
- pointed (doesn't contain a line)
- solid (nonempty interior), $\operatorname{int}(\mathbb{S}_+^{n\times n}) = \mathbb{S}_{++}^{n\times n}$

Product cones

The cone $\mathcal{K}$ can be

Linear: $x \geq 0$

Second-order: $x_0 \geq \|x\|_2$

Rotated second-order: $x_0 x_1 \geq \|x_{2:n}\|$, and $x_0 \geq 0$

Semidefinite: x is (can be assembled into) a symmetric, positive semidefinite matrix

or a product of these.

Example: $\mathcal{K} = \mathbb{S}_+^{n \times n} \times \mathbb{S}_+^{\ell \times \ell}$

Semidefinite optimization - general form

The unknown is a matrix:

$$\begin{aligned} \min\ & \mathrm{Tr}(CX) \\ & \mathrm{Tr}(A_i X) = b_i, i = 1, \dots, m \\ & X \succeq 0 \end{aligned} \qquad \begin{aligned} \max\ & b^T y \\ & \sum_{i=1}^m A_i y_i + S = C \\ & S \succeq 0 \end{aligned}$$

- C, X, S, A_i are $n \times n$ symmetric matrices
- $b, y \in \mathbb{R}^m$ are vectors

Semidefinite optimization - simplified notation

The unknown is a matrix:

$$\begin{aligned} \min\ C \bullet X \\ \mathcal{A}X = b \\ X \succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \max\ b^T y \\ \mathcal{A}^* y + S = C \\ S \succeq 0 \end{aligned}$$

- C, X, S are $n \times n$ symmetric matrices
- $b, y \in \mathbb{R}^m$ are vectors
- $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ is a linear operator

Weak duality:

$$C \bullet X \geq b^T y$$