

Modelling and optimization with cones

Imre Pólik, PhD

Lehigh University
Department of Industrial and Systems Engineering

January 13, 2009

Outline

- 1 Problem definition
 - General conic problem
 - Semidefinite optimization
 - Solving conic optimization
- 2 Applications
 - QP as SOCP
 - Quadratic/linear fractions
 - Inequalities with rational powers
 - Robust optimization
 - Eigenvalue optimization
 - Combinatorial optimization
 - Stability analysis
 - Structural design
 - Polynomial optimization
- 3 Software

Symmetric conic optimization in general

Outline

Problem definition

General conic problem

Semidefinite
optimizationSolving conic
optimization

Applications

Software

$$\min c^T x$$

$$Ax = b$$

$$x \in \mathcal{K}$$

$$\max b^T y$$

$$A^T y + s = c$$

$$s \in \mathcal{K},$$

The cone \mathcal{K} can be

Linear: $x \geq 0$

Second-order: $x_0 \geq \|x\|_2$

Rotated second-order: $x_0 x_1 \geq \|x_{2:n}\|$, and $x_0 \geq 0$

Semidefinite: x is (can be assembled into) a symmetric,
positive semidefinite matrix

or a product of these.

Semidefinite optimization - notations

The unknown is a matrix:

$$\min \operatorname{Tr}(CX)$$

$$\max b^T y$$

$$\operatorname{Tr}(A_i X) = b_i, i = 1, \dots, m$$

$$\sum_{i=1}^m A_i y_i + S = C$$

$$X \succeq 0$$

$$S \succeq 0$$

C, X, S, A_i are $n \times n$ symmetric matrices, $b, y \in \mathbb{R}^m$ vectors

Special structure: A_i, C can be low rank or sparse

Solving conic optimization

- Algorithm: mostly interior point methods
- Iterations
 - SDP: $\mathcal{O}(\sqrt{n})$
 - SOCP: $\mathcal{O}(\sqrt{\#\text{cones}})$
 - practice: $\approx 50 - 100$
- Cost of one iteration
 - SDP: $\mathcal{O}(mn^3 + m^2n^2 + m^3)$
 - SOCP: $\mathcal{O}(m^3 + \dots)$
 - much less for sparse data
- Problem sizes:
 - SDP: $m \leq 10000, n \leq 5000$
 - SOCP $m \leq 10000, k \leq 10000$
 - more with sparse data
- High accuracy

1 Problem definition

2 Applications

- QP as SOCP
- Quadratic/linear fractions
- Inequalities with rational powers
- Robust optimization
- Eigenvalue optimization
- Combinatorial optimization
- Stability analysis
- Structural design
- Polynomial optimization

3 Software

Quadratic optimization

$$\min x^T Qx + c^T x$$

$$Ax = b$$

$$x \geq 0$$

Cholesky factorization: $LL^T = Q$

SOCP formulation:

$$\min u_0$$

$$Lx - u = \frac{1}{2}L^{-1}c$$

$$Ax = b$$

$$x \geq 0$$

$$u_0 \geq \|u\|$$

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

$$\sum_{i=1}^r \frac{\|A_i x + b_i\|^2}{a_i^T x + \beta_i} \leq t$$

can be represented as

$$\sum_{i=1}^r u_i \leq t$$

$$w_i^T w_i \leq u_i v_i, \quad i = 1, \dots, r$$

$$w_i = A_i x + b_i, \quad i = 1, \dots, r$$

$$v_i = a_i^T x + \beta_i \geq 0, \quad i = 1, \dots, r$$

Inequalities with rational powers

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

$$x_1^{-5/6} x_2^{-1/3} x_3^{-1/2} \leq t$$

$$x_1, x_2, x_3 \geq 0$$

Integer powers:

$$1 \leq x_1^5 x_2^2 x_3^3 t^6$$

$$x_1, x_2, x_3 \geq 0$$

Equivalent formulation (rotated SOCP):

$$w_1^2 \leq x_1 x_3 \quad (1 \leq x_1^4 x_2^2 x_3^2 t^6 w_1^2)$$

$$w_2^2 \leq x_3 w_1 \quad (1 \leq x_1^4 x_2^2 t^6 w_2^4)$$

$$w_3^2 \leq x_2 t \quad (1 \leq x_1^4 t^4 w_3^4 w_2^4)$$

$$w_4^2 \leq w_2 w_3 \quad (1 \leq x_1^4 t^4 w_4^8)$$

$$w_5^2 \leq x_1 t \quad (1 \leq w_4^8 w_5^8)$$

$$1 \leq w_4 w_5$$

Not unique!

Linear optimization with uncertain data:

$$\begin{aligned} \min \quad & c^T x \\ & a_j^T x - b_j \geq 0, \forall j = 1, \dots, m, \end{aligned}$$

where

$$\left\{ \left(\begin{array}{c} a_j \\ -b_j \end{array} \right) = \left(\begin{array}{c} a_j^0 \\ -b_j^0 \end{array} \right) + Pu : u \in \mathbb{R}^m, \|u\| \leq 1 \right\},$$

Robust LP:

$$\left(\left(\begin{array}{c} a_j^0 \\ -b_j^0 \end{array} \right) + Pu \right)^T \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0, \forall u : \|u\| \leq 1,$$

SOCP form:

$$(a_j^0)^T x - b_j^0 \geq \left\| P^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|.$$

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

**Eigenvalue
optimization**Combinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Problem

Given matrices A_1, \dots, A_m find a nonnegative linear combination whose smallest eigenvalue is maximal.

SDP formulation

$$\begin{aligned} \max \quad & \lambda \\ \sum_{i=1}^m A_i y_i - \lambda I & \succeq 0 \\ y_i & \geq 0, \quad i = 1, \dots, m \end{aligned}$$

Needed for stability analysis

Relaxation of binary variables

SDP

Binary constraints: $x_i \in \{0, 1\}$

LP relaxation: $x_i \in [0, 1]$

Equivalent form:

$$z_i = 2x_i - 1$$

$$z_i^2 = 1 (\Leftrightarrow z_i = \pm 1)$$

Matrix form:

$$Z \succeq 0$$

$$\text{diag}(Z) = 1$$

$$\text{rank}(Z) = 1 (\Leftrightarrow Z = zz^T)$$

Graph partitioning

SDP

Given $2m$ points with weighted edges between them, divide them into 2 equal partitions such that the total weight of edges between the two partitions is minimized.

A : incidence matrix, A_{kl} is the weight of the kl edge

$y_{ij} = 1$: point i is in partition j

y_j : the indicator vector of partition j

$y_j^T A y_j$: $2 \times$ the weight of edges with both ends in partition j

$\text{Tr}(Y^T A Y)$: twice the total weight of uncut edges

$e^T A e$: twice total weight of edges

$$\min e^T A e - \text{Tr}(Y^T A Y)$$

Y is a partition matrix

SDP relaxation ($X = Y Y^T$) \Rightarrow

$$\min e^T A e - \text{Tr}(A X)$$

$$\text{diag}(X) = 1$$

$$X e = m$$

$$X \succeq 0$$

$$X \succeq 0$$

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Theorem (Lyapunov stability)

Trajectories of $\dot{x} = Ax$ converge to 0 if and only if there is a positive definite P such that $A^T P + PA$ is negative definite.

Theorem (Circle criterion)

Linear state-feedback system:

$$\dot{x} = Ax + Bw, \quad x(0) = x_0$$

$$v = Cx$$

$$\sigma(v, w) = (\beta v - w)^T (w - \alpha v) \geq 0, \quad (\text{sector constraint, } \alpha < \beta)$$

Equivalent condition for quadratic stability:

$$\begin{matrix} & & & P \succ 0 \\ \left(\begin{array}{cc} A^T P + PA - 2\beta\alpha C^T C & PB + (\beta + \alpha)C^T \\ B^T P + (\beta + \alpha)C & -2 \end{array} \right) & \prec & 0 & \end{matrix}$$

Structural design I

SDP

M : degrees of freedom, (number of joints)

$f \in \mathbb{R}^M$: loads on the joints

\mathcal{F} : set of loading scenarios

$w \in \mathbb{R}^M$: displacements of the joints

N : number of bars

$t \in \mathbb{R}^N$: volume of bars (\leftarrow design variables)

$Q(t) = \sum_{i=1}^N b_i b_i^T t_i$: energy matrix of the truss

$\frac{1}{2} w^T Q(t) w$: potential energy for displacement w

$f^T w - \frac{1}{2} w^T Q w$: compliance, maximized by w

Goal: for a given set of loads design a truss with minimum worst case compliance

Constraints: volumes of bars (individual and total), obstacles, maximum displacements

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural designPolynomial
optimization

Software

Design problem (multiload):

$$\min_t \left\{ \sup_{f,w} f^T w - \frac{1}{2} w^T Q(t) w \right\}$$

$$t_i \in [\underline{t}_i, \bar{t}_i] \quad (\text{lower/upper bound on volumes})$$

$$\sum_{i=1}^N t_i \leq v \quad (\text{total volume})$$

Semidefinite reformulation:

$$\min_t \tau$$

$$\begin{pmatrix} \tau & f_i^T \\ f_i & Q \end{pmatrix} \succeq 0, \quad i = 1, \dots, k$$

$$t_i \in [\underline{t}_i, \bar{t}_i] \quad (\text{lower/upper bound on volumes})$$

$$\sum_{i=1}^N t_i \leq v \quad (\text{total volume})$$

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Theorem

If $p(x) : \mathbb{R} \rightarrow \mathbb{R}$ is a univariate polynomial
 $p(x) \geq 0, \forall x \Leftrightarrow p(x)$ is a sum-of-squares (SOS)

Example

$$p(x) = x^6 - 5x^4 + 6x^3 + 8x^2 - 14x + 5 = \\ (x^2 + x - 1)^2 + (x^3 - 3x + 2)^2 \text{ is SOS}$$

Example

Converse is not true:

$$p(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \geq 0, \text{ but NOT SOS}$$

$$\min p(x) \Leftrightarrow \max t \quad \Leftrightarrow \quad \max t \\ p(x) - t \geq 0, \forall x \quad p(x) - t \text{ is SOS}$$

Polynomial optimization II

SDP

$$q = (1, x, x^2, \dots, x^n)$$

Square:

$$p(x) = \left(\sum_{i=0}^n u_i x^i \right)^2 = \left(\sum_{i=0}^n u_i q_i \right)^2 = (u^T q)^2 = q^T (u u^T) q$$

SOS: $q^T U q$, where U is PSD

$$u_{44} = 1$$

$$u_{34} + u_{43} = 0$$

$$u_{24} + u_{33} + u_{42} = -5$$

$$u_{14} + u_{23} + u_{32} + u_{41} = 6$$

$$u_{13} + u_{22} + u_{31} = 8$$

$$u_{21} + u_{12} = -14$$

$$u_{11} = 5$$

$$U \succeq 0$$

$$U = \begin{pmatrix} 5 & -7 & -1 & 2 \\ -7 & 10 & 1 & -3 \\ -1 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{pmatrix}$$

$$u^{(1)} = \begin{pmatrix} -1 & 1 & 1 & 0 \end{pmatrix}$$

$$u^{(2)} = \begin{pmatrix} 2 & -3 & 0 & 1 \end{pmatrix}$$

Extensions to more variables, interval constraints

Software: (Gloptipoly, SOSTools), Yalmip

Outline

Problem definition

Applications

QP as SOCP

Quadratic/linear
fractionsInequalities with
rational powers

Robust optimization

Eigenvalue
optimizationCombinatorial
optimization

Stability analysis

Structural design

Polynomial
optimization

Software

Software

- Solvers - mostly open source, last 10 years
 - SOCP: MOSEK, SeDuMi, LOQO, SDPT3, PENNON
 - SDP: CSDP, SDPA, SDPA, SeDuMi, SDPT3, PENSDP
- Modelling languages
 - No commercial support for SDP
 - GAMS: only SOCP
 - Yalmip: Matlab based, very powerful
 - CVX: Matlab, more academic

Modelling and optimization with cones

Imre Pólik, PhD

Lehigh University
Department of Industrial and Systems Engineering

January 13, 2009